

# Generalized Third-Derivative Block Hybrid Method for Solving Third-Order Differential Equations

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Received: 19:11:2025

Accepted: 26:03:2026

Published: 30:03:2026

## Abstract:

In many scientific and engineering applications, real-world phenomena are modeled using differential equations in order to describe, analyze and interpret physical processes. This study is designed to formulate the development of third-derivative block hybrid method for the direct numerical solution of third-order initial value problems. The method is formulated using interpolation and collocation techniques based on power series expansion, resulting in a block scheme that incorporates four off-step points for improved accuracy and computational efficiency. The mathematical properties of the proposed method including order, error constant, consistency, zero-stability, convergence and region of absolute stability are thoroughly investigated to ensure its reliability. Stability analysis using the Boundary Locus Method confirms that the new method possesses a satisfactory region of absolute stability suitable for stiff and non-stiff problems. The effectiveness of the method is demonstrated through four numerical experiments involving both highly stiff and non-stiff third-order linear problems. Comparative analysis with existing methods reveals that the proposed block hybrid method consistently produces numerical approximations that closely match the exact solution, outperforming several classical approaches in terms of accuracy and stability. The results confirm that the new method is robust, computationally efficient, and suitable for solving a wide class of higher-order ordinary differential equations.

**Keywords:** Third-order initial value problems; Block hybrid method; Boundary Locus Method; stiff and non-stiff; Ordinary differential equations

## 1. INTRODUCTION

The dynamic behavior of system is an interesting and important subject of study for scientists from different fields like engineering, economics, sciences, biology and so on. Differential equations can be classified as ordinary and partial. It also classified according to their order. The order of a differential equation is the highest derivative in the equation that is, the highest power of the highest derivative in a differential equation is the degree of the equation [1].

The solution of a differential equation is a value of the dependent variable in the equation that satisfies the equation at all points of the solution domain. The solution of differential equation at a point is the value of the dependent variable at that point. Differential equations along with a specified value of the unknown function at a given point in the domain of the solution are an initial value problem. This specified value is the initial condition. In many important cases of differential equations, analytic solutions are difficult to obtain and time consuming and result in numerical methods. Although numerical solutions are approximations, the error of approximation is often acceptable and gave birth to algorithms that are used to design computer simulated solutions.

This study examines the approximate solution of a third-order initial value problem expressed in the following form:

$$y'''(g) = f(g, y(g)), y(g_0) = y_0, y'(g_0) = y'_0, y''(g_0) = y''_0 \quad (1)$$

The method generates numerical solutions at one point at a time. Despite its usefulness, it is often limited by heavy computational demands, which can reduce the accuracy of the results through increased errors. To address these shortcomings, the block method was introduced by [2]. Earlier, Milne as cited in [3] had developed the idea of using a block of points to obtain starting values for predictor-corrector algorithms, a concept that was later expanded into a full numerical method.

In many scientific and engineering applications, real-world phenomena are modeled using differential equations to describe, analyze and interpret physical processes [3]. Numerous practical problems involve quantities whose behavior depends on the rate of change of one or more variables; these rates are represented through derivatives, which naturally give rise to differential equations. Traditionally, higher-order initial value problems are solved by converting them into equivalent systems of first-order ordinary differential equations [4, 5]. To address the limitations associated with this reduction process, direct methods are increasingly being considered. In this study, emphasis is placed on developing one-step hybrid block methods that utilize four off-step points for treating third order initial value problem. Accordingly, a new class of generalized one-step hybrid block methods with four off-step points for solving higher-order ordinary differential equations was

developed [6]. Block methods was introduced specifically for the purpose of generating accurate numerical approximations, following the approaches described by [7, 8].

Derivations of a one-step hybrid block method for directly solving fifth-order initial value problems of ordinary differential equations. This demonstrated that it is accurate, stable, consistent and equally convergent [9]. [10] developed the Generalized Runge-Kutta-Mohammed (GRKM) methods and used it to directly solve fifth-order ordinary differential equations (ODEs) without converting them into first-order systems. This approach offers two variants based on the level of derivative involvement.

Stiff differential equations are systems where certain components evolve much faster than others, requiring extremely small-time steps in numerical solvers for stability, even when the solution changes slowly. Non-stiff equations do not have this constraint. Stiff problems are often solved using implicit methods, while non-stiff problems are handled efficiently by explicit methods. Shampine and Thompson as reported in [11], the best way to detect stiffness is to try one of the solvers intended for non-stiff systems. If it is unsatisfactory, then the problem may be stiff and effective stiff solvers should be employed.

Furthermore, several researchers including [12-19] have developed block methods for obtaining direct solutions to higher-order ordinary differential equations. Their findings show that applying block techniques directly to higher-order equations achieves superior accuracy compared to first reducing the problems to systems of first-order equations. Inspired by these contributions, the present study focuses on constructing a one-step third-derivative block method for solving high-order initial value problems using a power series approach based on interpolation and collocation.

**2. Mathematical Formulation of the Method**

This section presents the construction of a third-derivative block hybrid method based on interpolation and collocation techniques for directly solving third-order initial value problems of ordinary differential equations of the form (1). A one-step block approach is employed, from which the corresponding block scheme is derived using the stated formulation.

$$y_{n+\xi} = \sum_{i=0}^2 \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^7 (\vartheta_{i\xi} f_{n+i} + \eta_{i\xi} g_{n+i}), \quad \xi = m, n, u, v, p, q, 1 \tag{2}$$

Obtain the first and second derivative schemes of the block method from

$$y_{n+\xi}^{(a)} = \sum_{i=0}^{2-\nu} \frac{(\xi h)^i}{i!} y_n^{(i+\nu)} + \sum_{i=0}^7 \varpi_{\xi i \nu} f_{n+i}, \quad \nu = 1_{(m, n, u, v, p, q, 1)}, \nu = 2_{(m, n, u, v, p, q, 1)} \tag{3}$$

$$\vartheta_{\xi i} = A^{-1}B \text{ and } \varpi_{\xi i \nu} = A^{-1}D \text{ where:}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & mh & nh & uh & vh & ph & qh & h \\ 0 & \frac{(mh)^2}{2!} & \frac{(nh)^2}{2!} & \frac{(uh)^2}{2!} & \frac{(vh)^2}{2!} & \frac{(ph)^2}{2!} & \frac{(qh)^2}{2!} & \frac{h^2}{2!} \\ 0 & \frac{(mh)^3}{3!} & \frac{(nh)^3}{3!} & \frac{(uh)^3}{3!} & \frac{(vh)^3}{3!} & \frac{(ph)^3}{3!} & \frac{(qh)^3}{3!} & \frac{h^3}{3!} \\ 0 & \frac{(mh)^4}{4!} & \frac{(nh)^4}{4!} & \frac{(uh)^4}{4!} & \frac{(vh)^4}{4!} & \frac{(ph)^4}{4!} & \frac{(qh)^4}{4!} & \frac{h^4}{4!} \\ 0 & \frac{(mh)^5}{5!} & \frac{(nh)^5}{5!} & \frac{(uh)^5}{5!} & \frac{(vh)^5}{5!} & \frac{(ph)^5}{5!} & \frac{(qh)^5}{5!} & \frac{h^5}{5!} \\ 0 & \frac{(mh)^6}{6!} & \frac{(nh)^6}{6!} & \frac{(uh)^6}{6!} & \frac{(vh)^6}{6!} & \frac{(ph)^6}{6!} & \frac{(qh)^6}{6!} & \frac{h^6}{6!} \\ 0 & \frac{(mh)^7}{7!} & \frac{(nh)^7}{7!} & \frac{(uh)^7}{7!} & \frac{(vh)^7}{7!} & \frac{(ph)^7}{7!} & \frac{(qh)^7}{7!} & \frac{h^7}{7!} \end{pmatrix}, B = \begin{pmatrix} \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \end{pmatrix}, D = \begin{pmatrix} \frac{(\xi h)^{3-b}}{(3-b)!} \\ \frac{(\xi h)^{4-b}}{(4-b)!} \\ \frac{(\xi h)^{5-b}}{(5-b)!} \\ \frac{(\xi h)^{6-b}}{(6-b)!} \\ \frac{(\xi h)^{7-b}}{(7-b)!} \\ \frac{(\xi h)^{8-b}}{(8-b)!} \\ \frac{(\xi h)^{9-b}}{(9-b)!} \\ \frac{(\xi h)^{10-b}}{(10-b)!} \end{pmatrix}$$

The implementation of (2) and (3), can also be written in the following form

$$\left. \begin{aligned} y_{n+m} &= y_n + mh y'_n + \frac{(mh)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+n} &= y_n + nh y'_n + \frac{(nh)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+u} &= y_n + uh y'_n + \frac{(uh)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+v} &= y_n + vh y'_n + \frac{(vh)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+p} &= y_n + ph y'_n + \frac{(ph)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+q} &= y_n + qh y'_n + \frac{(qh)^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \\ y_{n+1} &= y_n + h y'_n + \frac{h^2}{2!} y''_n + [\vartheta_{01} f_n + \vartheta_{02} f_{n+m} + \vartheta_{03} f_{n+n} + \vartheta_{04} f_{n+u} + \vartheta_{05} f_{n+v} + \vartheta_{06} f_{n+p} + \vartheta_{07} f_{n+q} + \vartheta_{08} f_{n+1}] \end{aligned} \right\} \tag{4}$$

$$\left. \begin{aligned} y'_{n+m} &= y'_n + mh y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+n} &= y'_n + nh y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+u} &= y'_n + uh y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+v} &= y'_n + vh y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+p} &= y'_n + ph y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+q} &= y'_n + qh y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \\ y'_{n+1} &= y'_n + h y''_n + [\varpi_{11} f_n + \varpi_{12} f_{n+m} + \varpi_{13} f_{n+n} + \varpi_{14} f_{n+u} + \varpi_{15} f_{n+v} + \varpi_{16} f_{n+p} + \varpi_{17} f_{n+q} + \varpi_{18} f_{n+1}] \end{aligned} \right\} \tag{5}$$

$$\left. \begin{aligned} y''_{n+m} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+n} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+u} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+v} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+p} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+q} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \\ y''_{n+1} &= y''_n + [\varpi_{21} f_n + \varpi_{22} f_{n+m} + \varpi_{23} f_{n+n} + \varpi_{24} f_{n+u} + \varpi_{25} f_{n+v} + \varpi_{26} f_{n+p} + \varpi_{27} f_{n+q} + \varpi_{28} f_{n+1}] \end{aligned} \right\} \tag{6}$$

The values unknown coefficients  $\vartheta, \varpi$  are given in appendix.

**3. Investigating the Properties of the Block Method**

The analysis of the basic properties of the method derived, which include order, error constant, consistency, zero-stable and convergence.

**Order and error constant of the block method**

According to [18], the order and error constant of the block method

$$y^{(m)}(x_n + ah) = y^{(m)}(x_n) + ah y^{(m+1)}(x_n) + \frac{(ah)^2}{2!} y^{(m+2)}(x_n) + \frac{(ah)^3}{3!} y^{(m+3)}(x_n) + \dots \tag{7}$$

where  $y^{(m)}x_n = \left. \frac{d^m y}{dx^m} \right|_{x=x_n}, m = 1, 2, \dots$

Using the linear operator

$$L[y(x); h] = \sum_{j=0}^1 \alpha_j y_{n+j} - \sum_{j=0}^1 \beta_j f_{n+j} + \sum_{j=0}^1 \gamma_j f'_{n+j} \tag{8}$$

Expanded (8) using Taylor series expansions about  $x_n$  and comparing the coefficient  $h$  and the method is said to be of order  $p$  if  $C_0 = C_1 = \dots = C_{p+1} = 0, C_{p+2} = 0, C_{p+3} \neq 0$  and  $C_{p+3}$  is the error constant. Therefore the order and error constant of third derivative block method are  $p = [6, 6, 6, 6, 6, 6]$  and

$$C_{p+3} = [-1.3619e-11, -1.0520e-11, -1.2003e-11, -1.0723e-11, -1.2702e-11, -6.4286e-12, -8.0743e-09]^T$$

**Consistency of the block method**

According to [3], the block method is said to be consistent if the order is greater than or equal to one i.e.  $p \geq 1$ .

**Zero Stability of the block method**

The block method is said to be zero-stable, if the roots  $z_s, s = 1, 2, \dots, k$  of the first characteristics polynomial  $\rho(z)$  defined by  $\rho(z) = \det(zA^{(0)} - E)$  satisfies  $|z_s| \leq 1$  and every root satisfies  $|z_s| = 1$  have multiplicity not exceeding the order of the differential equation [see, 17]. To analyze the block method for zero stability, the roots of the first characteristic polynomial:

$$\rho(z) = zI_7 - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Solving for  $z$  in (9) gives  $\rho(z) = z^6 - z^5$ .

**Convergence of the block method**

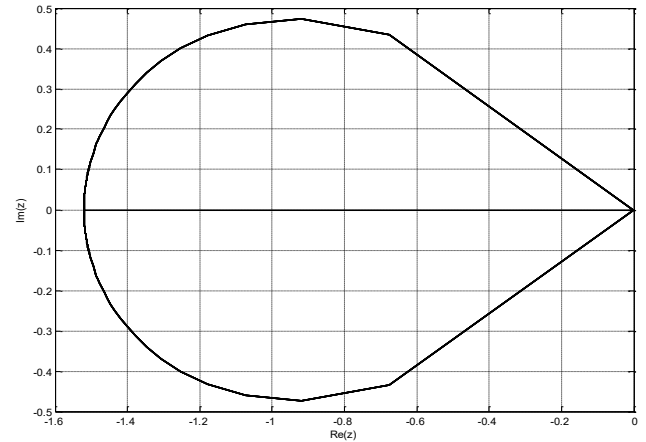
According to [3], the block method is to be convergent if it is consistent and zero-stable. Hence the block method is convergent.

**Region of Absolute Stability of the block method**

According to [3], the Boundary Locus Method were used on the new method to obtain the stability polynomial as:

$$\bar{h}(w) = \left(-\frac{1}{241920} w^5 - \frac{1}{241920} w^6\right) h^{12} + \left(-\frac{11}{103680} w^5 - \frac{11}{103680} w^6\right) h^{10} + \left(-\frac{19}{10368} w^5 + \frac{7}{4320} w^6\right) h^8 + \left(-\frac{29}{1728} w^5 - \frac{29}{1728} w^6\right) h^6 + \left(-\frac{65}{432} w^4 + \frac{101}{864} w^5\right) h^4 + \left(-\frac{1}{2} w^5 - \frac{1}{2} w^6\right) h^2 - 2w^4 + w^5 \tag{10}$$

Using the stability polynomial (10), the region of absolute stability of the new method is shown in Figure 1 as:



**Figure 1:** Stability region of the new method. The stability region obtained in Figure 1 is *A – stable*.

**Procedure for solving Higher order ODEs**

Rewrite the third order ODE as a system of first order ODEs.

For the  $n$ th order ODE:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

$$\text{set } y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$$

The result of this substitution will be an equivalent system of  $n$  first order ODEs

$$y'_1 = y_2$$

$$y'_2 = y_3$$

$\vdots$

$$y'_n = f(x, y_1, y_2, \dots, y_n)$$

For example consider the third order ODE

$$y''' = f(x, y, y', y'')$$

$$\text{set } y_1 = y, y_2 = y' \text{ and } y_3 = y''$$

$$\text{This implies that } y'_1 = y' = y_2 \text{ and } y'_2 = y'' = y_3$$

Hence, we obtain the system of first order ODEs  $y'_3 = f(x, y_1, y_2, y_3)$

**4. Numerical Example of the block method**

In this section, the numerical examples considered include various stiff and non-stiff third order differential equations that were presented. The numerical applications of propose method were presented through the MATLAB by calculating the absolute error using  $|y(g) - y|$  where  $y(g)$  is exact solution and  $y$  is approximate solution. Hence, the illustration in tabular and graphical forms to measure the effectiveness, accuracy and usefulness of propose method were pointed out.

**Problem 1:** Consider the highly non- stiff third order linear problem of the form:

$$y''' - 3\sin(g) = 0, \quad y(0) = 1, y'(0) = 0, y''(0) = -2 \quad (11)$$

with the exact solution given by

$$y(g) = 3\cos(g) + \frac{g^2}{2} - 2 \quad (12)$$

Source: [8, 18, 20].

Problem 2: Consider the highly stiff third order linear problem  $y''' + \exp(g) = 0, y(0) = 1, y'(0) = -1, y''(0) = 3, h = 0.1$  (13)

with the exact solution given by

$$y(g) = 2g^2 - \exp(g) + 2 \quad (14)$$

Source [21, 22, 23].

Problem 3: Consider the highly stiff third order linear problem  $y''' + y' = 0, y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.1$  (15)

with the exact solution given by

$$y(g) = 2(1 - \cos g) + \sin g \quad (16)$$

Source [18, 24, 25].

Problem 4: Consider the highly stiff third order linear problem  $y''' + y = 0, y(0) = 1, y'(0) = -1, y''(0) = 1, h = 0.1$  (17)

with the exact solution given by

$$y(g) = \exp(-g) \quad (18)$$

Source [7, 24, 26].

The following notations shall be used in Tables.

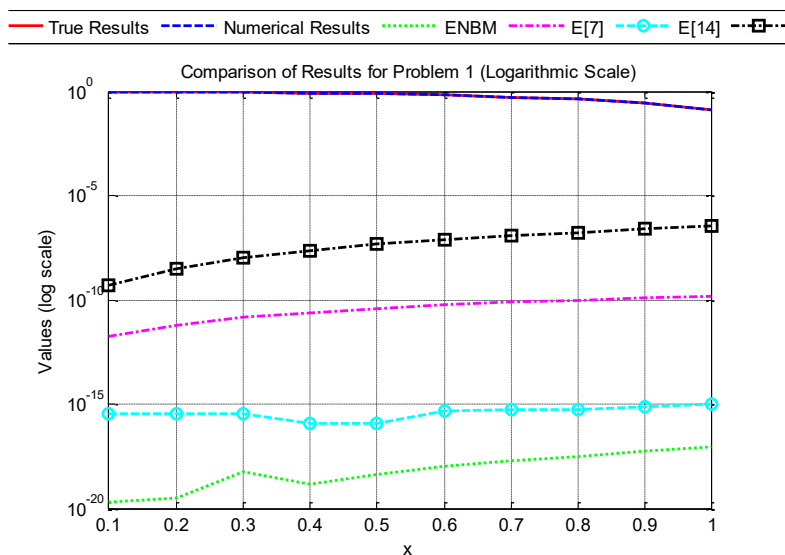
Notations	Meaning
<b>G</b>	<b>Independent variable used</b>
ENBM	Error in new block method
E[8]	Error in [8]
E[17]	Error in [17]
E[20]	Error in [20]
E[21]	Error in [21]
E[22]	Error in [22]
E[23]	Error in [23]
E[24]	Error in [24]
E[25]	Error in [25]
E[26]	Error in [26]

### Discussion of Results

The results in **Table 1** and its corresponding curve (**Figure 2**) present the solution of a highly non-stiff third-order linear problem.

**Table 1:** Showing the results for problem 1

g	True Results	Numerical results	ENBM	E[8]	E[17]	E[20]
0.1	0.990012495834 07729830	0.9900124958 3407729828	2.0000e-20	1.7282e-12	3.3307e-16	4.8906e-10
0.2	0.960199733523 72489340	0.9601997335 2372489337	3.0000e-20	6.3179e-12	3.3307e-16	3.2663e-09
0.3	0.911009467376 81805890	0.9110094673 7681805896	6.0000e-19	1.4295e-11	3.3307e-16	1.0296e-08
0.4	0.843182982008 65524840	0.8431829820 0865524855	1.5000e-19	2.5020e-11	1.1102e-16	2.3509e-08
0.5	0.757747685671 11814840	0.7577476856 7111814879	3.9000e-19	3.8928e-11	1.1102e-16	4.4764e-08
0.6	0.656006844729 03489170	0.6560068447 2903489271	1.0100e-18	5.5360e-11	4.4409e-16	7.5847e-08
0.7	0.539526561853 46527880	0.5395265618 5346528066	1.8600e-18	7.4644e-11	5.5511e-16	1.1844e-07
0.8	0.410120128041 49626280	0.4101201280 4149626607	3.2700e-18	9.6128e-11	5.5511e-16	1.7411e-07
0.9	0.269829904811 99336940	0.2698299048 1199337486	5.4600e-18	1.2002e-10	7.2164e-16	2.4429e-07
1.0	0.120906917604 41915220	0.1209069176 0441916055	8.3500e-18	1.4570e-10	1.0547e-15	3.3028e-07



**Figure 2:** Curve of Table 1

The table shows both the exact (True) results and the numerical approximations obtained using different methods. From the data, the numerical results closely match the true values throughout the interval. The curve illustrates that all methods capture the trend of the solution accurately, with the newly proposed block method closely following the exact solution, demonstrating its reliability in solving non-stiff problems.

**Table 2:** Showing the results for problem 2

g	True Results	Numerical results	ENB M	E[21]	[22]	E[23]
0.1	0.91482908192 43523752	0.9148290 819243523	0.0000e-00	1.1102e-14	1.8241e-13	1.6209e-14
0.2	0.85859724183 98301661	0.8585972 418398301	8.0000e-20	1.6076e-13	1.6708e-12	6.6058e-14
0.3	0.83014119242 39968960	0.8301411 924239968	2.4000e-19	6.3105e-13	6.0014e-12	1.5277e-13
0.4	0.82817530235 87296822	0.8281753 023587296	7.4000e-19	1.6232e-12	1.4860e-11	2.7955e-13
0.5	0.85127872929 98718532	0.8512787 292998718	1.5800e-18	3.3591e-12	3.0121e-11	4.5020e-13
0.6	0.89788119960 94910251	0.8978811 996094910	2.8000e-18	6.0841e-12	5.3842e-11	6.6835e-13
0.7	0.96624729252 95234784	0.9662472 925295234	4.7800e-18	1.0070e-11	8.8316e-11	9.3925e-13
0.8	1.05445907150 75323954	1.0544590 715075323	7.3000e-18	1.5616e-11	1.3606e-10	1.2670e-12
0.9	1.16039688884 30503362	1.1603968 888430503	1.1000e-17	2.3054e-11	1.9987e-10	1.6578e-12
1.0	1.28171817154 09547646	1.2817181 715409547	1.5700e-17	3.2750e-11	2.8281e-10	2.1174e-12

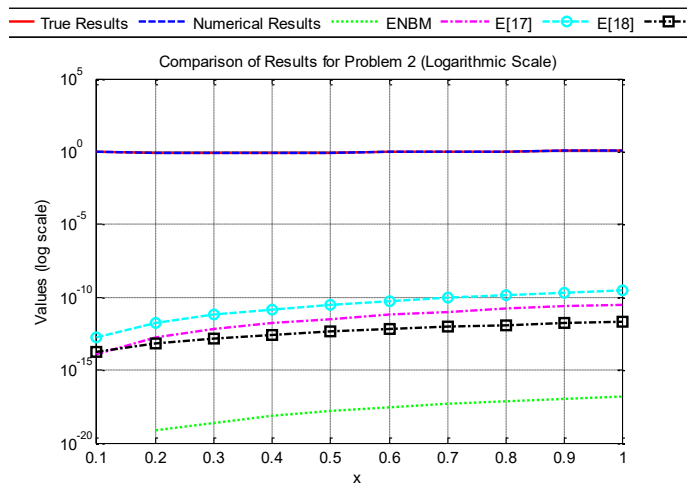


Figure 3: Curve of Table 2

Table 3: Showing the results for problem 3

$g$	True Results	Numerical results	ENB M	E[24]	E[20]	E[25]
0.1	0.109825086090	0.10982508609077	0.000	1.117	3.747	2.498
0.2	0.238536175112	0.23853617511257	0e-00	7e-10	0e-16	0e-16
0.3	0.384847228410	0.38484722841012	5.000	3.277	1.387	8.326
0.4	0.547296354302	0.54729635430288	0e-20	5e-09	8e-15	7e-16
0.5	0.724260414823	0.72426041482345	1.620	1.624	1.554	4.440
0.6	0.913971243575	0.91397124357567	2.950	2.891	1.998	1.110
0.7	1.114533312668	1.11453331266871	5.000	4.712	2.886	4.440
0.8	1.323942672205	1.32394267220519	7.800	7.198	4.440	1.332
0.9	1.540106973086	1.54010697308615	1.150	1.045	3.552	4.440
1.0	1.760866373071	1.76086637307161	1.620	1.459	5.329	2.220

Table 4: Showing the results for problem 4

	True Results	Numerical results	ENBM	E[25]	E[26]	E[24]
0.1	0.904837418035959573	0.90483741803595957316	0.0000e-00	2.1824e-12	3.5442e-11	6.0396e-14
0.2	0.818730753077981858	0.81873075307798185862	5.0000e-20	4.0342e-12	5.3218e-11	4.7196e-13
0.3	0.740818220681717866	0.74081822068171786584	2.3000e-19	5.3888e-12	1.3712e-09	1.5618e-12
0.4	0.670320046035639300	0.67032004603563930016	5.8000e-19	6.4744e-11	8.4438e-09	3.6204e-12
0.5	0.606530659712633423	0.60653065971263342241	1.1900e-18	5.9078e-11	2.9737e-08	6.9109e-12
0.6	0.548811636094026432	0.54881163609402643053	2.1000e-18	1.1672e-11	7.8367e-08	1.6667e-11
0.7	0.496585303791409514	0.49658530379140951135	3.3500e-18	2.2523e-11	4.3298e-05	1.8098e-11
0.8	0.449328964117221591	0.44932896411722158642	5.0100e-18	3.9121e-11	9.6716e-05	2.6384e-11
0.9	0.406569659740599111	0.40656965974059910480	7.0800e-18	6.1177e-11	1.6111e-04	3.6683e-11
1.0	0.367879441171442321	0.36787944117144231197	9.6300e-18	4.9220e-11	2.3725e-04	4.9122e-11

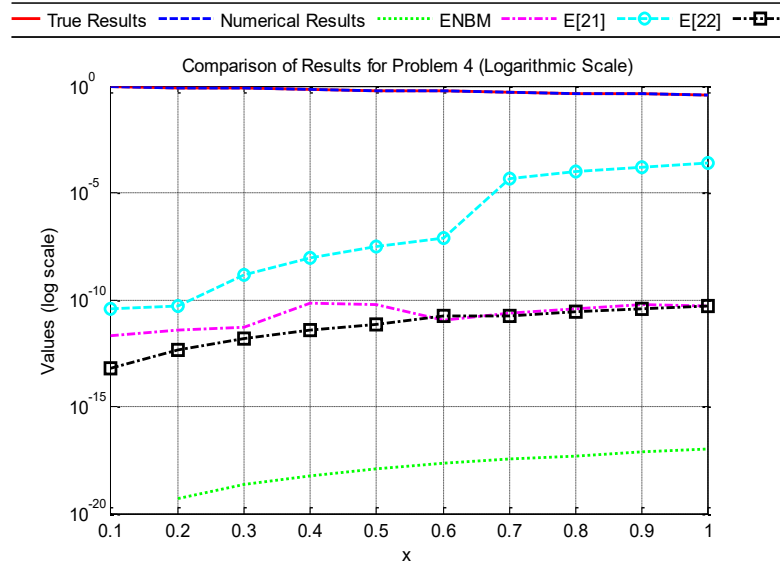


Figure 5: Curve of Table 4

Table 2 and Figure 3 display the results for a highly stiff third-order linear problem. Despite the stiffness, the numerical solutions produced by all methods closely follow the true solution across the range of values. The plot highlights that while all methods approximate the solution well, the new block method maintains excellent consistency and stability, especially for larger values of the independent variable  $x$ , indicating its suitability for stiff problems.

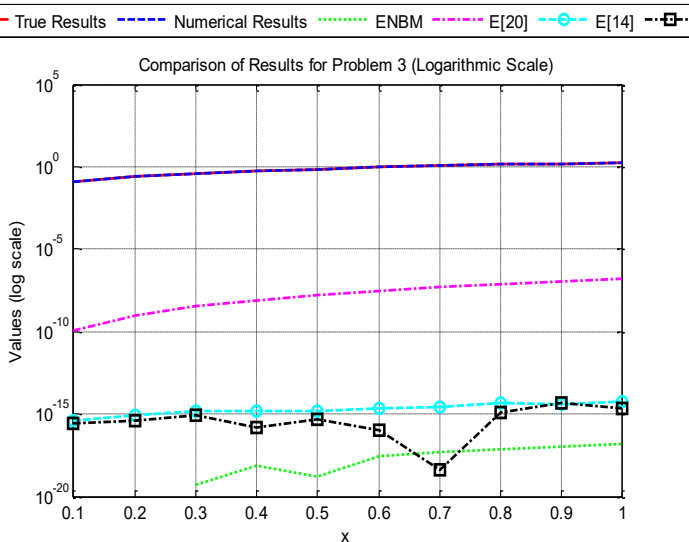


Figure 4: Curve of Table 3

For **Problem 3** (Table 3 and Figure 4), another stiff third-order linear problem is considered. The numerical results align almost perfectly with the exact solution. The curve demonstrates that all methods reproduce the overall shape and trend of the solution, but the new block method consistently follows the true solution more precisely across the entire interval. This shows the method's ability to handle stiffness while preserving the accuracy of the computed solution.

**Table 4** and **Figure 5** extend the analysis to a further stiff third-order linear problem. The numerical approximations from all methods track the exact solution closely, and the curves clearly show that the new block method stays very close to the true solution throughout. Across all four problems, the new block method demonstrates robustness, stability, and reliability in capturing the behavior of both stiff and non-stiff third-order linear problems, while all methods generally reproduce the correct solution trend.

### Summary and Conclusion

The study develops a one-step third-derivative block hybrid method designed to solve third-order initial value problems directly without reducing them to systems of first-order equations. Using interpolation and collocation techniques anchored on power series expansion, the method constructs a block scheme incorporating four off-step points to enhance accuracy and computational efficiency. The mathematical properties-including order, error constant, consistency, zero-stability, convergence, and absolute stability-were thoroughly investigated, confirming that the method meets all necessary theoretical requirements. Stability analysis using the Boundary Locus Method further demonstrates that the proposed scheme possesses a satisfactory stability region suitable for both stiff and non-stiff differential equations.

The results obtained from four numerical experiments show that the new block hybrid method performs exceptionally well compared to several existing methods in the literature. Across both stiff and non-stiff third-order linear problems, the numerical approximations closely match the exact solutions, with significantly reduced error values. This demonstrates the method's robustness, reliability, and computational advantage. Overall, the proposed third-derivative block hybrid method proves effective for solving higher-order ordinary differential equations and stands as a competitive alternative to traditional block and hybrid numerical approaches.

### Authorship Contribution Statement

**Raji, Musiliu Tayo:**-Original draft Conceptualization, Methodology, Rewriting and Editing

**Bello Kareem Akanbi:**- Methodology, data preparation and Validation

**Fasasi Yekeen Adegoke:**- Computation and Visualization and Proofreading

### Declaration of conflict of interests

The authors declare no conflict of interest.

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### Appendix

$$y_{n+\frac{1}{10}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{281833h^3}{3024000000}, \frac{1457h^3}{9720000}, -\frac{159931h^3}{967680000}, \frac{319h^3}{2016000}, -\frac{4583h^3}{45360000}, \frac{28871h^3}{756000000}, -\frac{127h^3}{19353600}, \frac{4001h^3}{217728000000} \right)$$

$$y_{n+\frac{1}{5}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{67477h^3}{141750000}, \frac{227h^3}{170100}, -\frac{377h^3}{360000}, \frac{719h^3}{708750}, -\frac{3667h^3}{5670000}, \frac{961h^3}{3937500}, -\frac{19h^3}{453600}, \frac{199h^3}{1701000000} \right)$$

$$y_{n+\frac{3}{10}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{32373h^3}{28000000}, \frac{9h^3}{2240}, -\frac{74439h^3}{35840000}, \frac{2799h^3}{1120000}, -\frac{1431h^3}{896000}, \frac{16929h^3}{28000000}, -\frac{1863h^3}{17920000}, \frac{261h^3}{896000000} \right)$$

$$y_{n+\frac{2}{5}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{2101h^3}{984375}, \frac{1744h^3}{212625}, -\frac{131h^3}{47250}, \frac{608h^3}{118125}, -\frac{151h^3}{50625}, \frac{3344h^3}{2953125}, -\frac{23h^3}{118125}, \frac{29h^3}{53156250} \right)$$

$$y_{n+\frac{1}{2}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{9899h^3}{2903040}, \frac{3025h^3}{217728}, -\frac{1625h^3}{516096}, \frac{1375h^3}{145152}, -\frac{625h^3}{145152}, \frac{7h^3}{3840}, -\frac{725h^3}{2322432}, \frac{61h^3}{69672960} \right)$$

$$y_{n+\frac{3}{5}}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{8721h^3}{1750000}, \frac{369h^3}{17500}, -\frac{891h^3}{280000}, \frac{27h^3}{1750}, -\frac{351h^3}{70000}, \frac{1377h^3}{437500}, -\frac{9h^3}{20000}, \frac{9h^3}{7000000} \right)$$

$$y_{n+1}(\vartheta_{01}, \vartheta_{02}, \vartheta_{03}, \vartheta_{04}, \vartheta_{05}, \vartheta_{06}, \vartheta_{07}, \vartheta_{08}) = \left( \frac{299h^3}{15120}, \frac{155h^3}{6804}, \frac{1675h^3}{12096}, -\frac{25h^3}{126}, \frac{2525h^3}{9072}, -\frac{617h^3}{3780}, \frac{415h^3}{6048}, \frac{179h^3}{544320} \right)$$

$$\begin{aligned}
 & y'_{n+\frac{1}{10}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{1049479h^2}{453600000}, \frac{197687h^2}{38880000}, \frac{248029h^2}{48384000}, \frac{251h^2}{51840}, \frac{13943h^2}{4536000}, \frac{175109h^2}{151200000}, \frac{72067h^2}{362880000}, \frac{6031h^2}{108864000000} \right) \\
 & y'_{n+\frac{1}{5}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{37697h^2}{7087500}, \frac{20458h^2}{1063125}, \frac{1193h^2}{108000}, \frac{824h^2}{70875}, \frac{17h^2}{2268}, \frac{1678h^2}{590625}, \frac{2771h^2}{5670000}, \frac{233h^2}{170100000} \right) \\
 & y'_{n+\frac{3}{10}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{92859h^2}{11200000}, \frac{38469h^2}{1120000}, \frac{783h^2}{89600}, \frac{2151h^2}{112000}, \frac{5283h^2}{448000}, \frac{24921h^2}{5600000}, \frac{1713h^2}{2240000}, \frac{3h^2}{1400000} \right) \\
 & y'_{n+\frac{2}{5}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{19966h^2}{1771875}, \frac{52496h^2}{1063125}, \frac{253h^2}{47250}, \frac{352h^2}{10125}, \frac{152h^2}{10125}, \frac{3536h^2}{590625}, \frac{367h^2}{354375}, \frac{31h^2}{10631250} \right) \\
 & y'_{n+\frac{1}{2}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{517h^2}{36288}, \frac{28045h^2}{435456}, \frac{775h^2}{387072}, \frac{3725h^2}{72576}, \frac{1475h^2}{145152}, \frac{59h^2}{6912}, \frac{785h^2}{580608}, \frac{13h^2}{3483648} \right) \\
 & y'_{n+\frac{3}{5}}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{753h^2}{43750}, \frac{348h^2}{4375}, \frac{27h^2}{28000}, \frac{12h^2}{175}, \frac{9h^2}{1750}, \frac{432h^2}{21875}, \frac{9h^2}{100000}, \frac{3h^2}{700000} \right) \\
 & y'_{n+1}(\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}, \varpi_{17}, \varpi_{18}) = \\
 & \left( \frac{215h^2}{2268}, \frac{610h^2}{1701}, \frac{9925h^2}{6048}, \frac{1600h^2}{567}, \frac{7325h^2}{2268}, \frac{374h^2}{189}, \frac{6245h^2}{9072}, \frac{293h^2}{54432} \right) \\
 \\
 & y''_{n+\frac{1}{10}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{74423h}{2419200}, \frac{204961h}{1814400}, \frac{98639h}{1075200}, \frac{3661h}{43200}, \frac{128969h}{2419200}, \frac{1343h}{67200}, \frac{16529h}{4838400}, \frac{11h}{1161216} \right) \\
 & y''_{n+\frac{1}{5}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{1861h}{63000}, \frac{2171h}{14175}, \frac{43h}{4200}, \frac{82h}{1575}, \frac{1367h}{37800}, \frac{37h}{2625}, \frac{31h}{12600}, \frac{h}{141750} \right) \\
 & y''_{n+\frac{3}{10}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{2677h}{89600}, \frac{669h}{4480}, \frac{13851h}{358400}, \frac{1313h}{11200}, \frac{4491h}{89600}, \frac{81h}{4480}, \frac{547h}{179200}, \frac{3h}{358400} \right) \\
 & y''_{n+\frac{1}{2}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{701h}{23625}, \frac{2144h}{14175}, \frac{31h}{1050}, \frac{832h}{4725}, \frac{17h}{4725}, \frac{32h}{2625}, \frac{11h}{4725}, \frac{h}{141750} \right) \\
 & y''_{n+\frac{2}{5}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{965h}{32256}, \frac{10825h}{72567}, \frac{1625h}{43008}, \frac{625h}{4032}, \frac{7375h}{96768}, \frac{151h}{2688}, \frac{275h}{64512}, \frac{11h}{1161216} \right) \\
 & y''_{n+\frac{3}{5}}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{41h}{1400}, \frac{27h}{175}, \frac{27h}{1400}, \frac{34h}{175}, \frac{27h}{1400}, \frac{27h}{175}, \frac{41h}{1400}, 0 \right) \\
 & y''_{n+1}(\varpi_{21}, \varpi_{22}, \varpi_{23}, \varpi_{24}, \varpi_{25}, \varpi_{26}, \varpi_{27}, \varpi_{28}) = \\
 & \left( \frac{827h}{1512}, \frac{2125h}{567}, \frac{2125h}{168}, \frac{4250h}{189}, \frac{36625h}{1512}, \frac{311h}{21}, \frac{6875h}{1512}, \frac{89h}{1134} \right)
 \end{aligned}$$