Bi-Univalency of a Generalised Distribution Series Convoluted with Beta Function and Poisson Distribution Series via Bells Number

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Abstract:
In this study, the number of equivalence relations on a set of n elements is given by the n-th Bell number (Bn). These numbers represent all possible partitions of a set. By using Beta and Poisson distribution series we applied convolution principle in order to investigate coefficient estimates. The researchers focused on the relationship between Bell numbers and the bi-univalency of a generalized distribution series that combines the Poisson distribution series and the Beta function. To achieve the results, the initial bounds on coefficients for the specified classes of functions will be employed to derive the well-known Fekete-Szegö inequalities. These findings signify a fresh contribution to the realm of Geometric Function Theory (GFT) as there has been no existing literature that discusses the convolution involving both the Beta function and the Poisson distribution series.

Keywords: **univalent function, analytic function, beta function, Poisson distribution, Bell number.**

1. Introduction

REPRESENT by H the function of the form:

$$t(z) = z + \gamma_2 z^2 + \gamma_3 z^3 + \gamma_4 z^4 + \cdots \quad (1)$$

which are the class of analytic function in the open unit disk $\{E = z: z \in C \text{ and } |z| < 1\}$ and normalized with the condition $t(0) = 0$ and $t'(0) - 1 = 0$.

Let S be a subset of H made up of univalent functions that are defined inside the unit disk E = {z: z∈C and |z|<1}, adhering to the previously mentioned conditions. Equation (1) has served as a fundamental principle for other subclasses of functions such as bounded turning points, starlike functions, convex functions, spiral-like functions, and close-to-convex functions that meet the specified representations.

$\mathrm{Re}\{t'(z)\} > 0$, $\mathrm{Re}\left\{\frac{zt'(z)}{t(z)}\right\} > 0$, $\mathrm{Re}\left\{\frac{zt''(z)}{t'(z)}\right\} > 0$, $\mathrm{Re}\left\{e^{i\theta}\frac{zt'(z)}{t(z)}\right\} > 0$, $\mathrm{Re}\left(\frac{zt'(z)}{g(z)}\right) > 0$ and just to mention but few. The subclasses previously referenced have been explored by various researchers from different viewpoints, resulting in numerous published findings. Refer to details [2, 3, 4, 12, 22].

In the open unit disk E, t and r be two analytic functions. We state that t is subordinate to r, expressed as:

$$t(z) \prec r(z), \quad z \in \mathbb{E},$$

If a Schwarz function ω(z), analytic in E exists, such that $\omega(0) = 0$, $|\omega(z)| < 1$, and

$$t(z) = r(\omega(z)) \quad z \in \mathbb{E} \quad (2)$$

In particular, the criteria if r is univalent in E, $t(0) = r(0)$, $t(\mathbb{E}) \subset r(\mathbb{E})$ are identical to the above statement.

Let

$$t(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad r(z) = \sum_{n=0}^{\infty} b_n z^n$$

be two of E's analytical functions. The definition of their Hadamard product, or convolution, is

$$(t * r)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (3)$$

The probability behaviour of coefficient sequences and its connection to analytic function features, the generalized distribution series has become more well-known in Geometric Function Theory (GFT) because of its usefulness. The concept was initially formalized by Porwal [30], who established a connection between generalized discrete distributions and analytic generating functions.

This investigation was achieved by seeking $\{a_n\}_{n=0}^{\infty}$ to be a series of real, non-negative values that

$$G = \sum_{n=0}^{\infty} a_n \quad (4)$$

has a convergent nature. The associated probability mass function takes the form:

$$P(n) = \frac{a_n}{G}, \quad n \in N \cup \{0\} \quad (5)$$

A legitimate probability distribution is formed by this function because for P(n)≥0 and

$$\sum_{n=0}^{\infty} P(n) = 1$$

The sequence's matching generating function is provided by

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (6)$$

which, given appropriate circumstances, converges at x=1 and converges for |x|<1. This distribution yielded the following mean μ_1' and second moment around the origin μ_2'

$$\mu_1' = \frac{\phi'(1)}{G}, \quad \mu_2' = \frac{1}{G}[\phi''(1) + \phi'(1)].$$

Hence, the variance is given by:

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{1}{G}[\phi''(1) + \phi'(1)] - \left(\frac{\phi'(1)}{G}\right)^2.$$

The usage of generalized distribution series has been expanded by researchers such as Olatunji et al. [24], Olatunji et al. [23], Oladipo [19, 20, 21], and Porwal [26, 28] to investigate statistical properties of analytic functions and their geometric subclasses. Some GFT experts have recently shown a great deal of interest in polynomials whose coefficients represent probabilities of the generalized distribution. This success hinges on the special characteristics of the polynomials, which have the following form:

$$P_\sigma = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{G} z^n \tag{7}$$

Where $G = \sum_{n=0}^{\infty} a_n$, $a_n \geq 0$ for all $n \in \mathbb{N}$.

Differential equations and other scientific and engineering fields can make use of the Euler integral of the first sort, generally known as the beta function. We will look at this function to leverage on some of its special features.

Let B(m,n) represent the form's beta function.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{8}$$

For m > 0 and n > 0, it converges. For equation (8) above, we have the following three exceptional cases:

a. When the number m is positive,
b. when n is an integer that is positive, and
c. when the integers m and n are both positive.

The symmetric quality of the Beta function can be expressed using the Gamma function, considering instance (c) above.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = B(n,m). \tag{9}$$

Using Ratio test in equation (9), let

$$a_k = \frac{(m-1)!\,(\mathrm{k}-1)!}{(m+\mathrm{k}-1)!} \text{ and } a_{\mathrm{k}+1} = \frac{\mathrm{k}!\,(m-1)!}{(m+\mathrm{k})!}$$

then

$$\frac{a_{k+1}}{a_k} = \frac{\mathrm{k}!\,(m-1)!}{(m+\mathrm{k})!} \cdot \frac{(m+\mathrm{k}-1)!}{(m-1)!\,(\mathrm{k}-1)!} = \frac{\mathrm{k}}{m+\mathrm{k}}$$

converges if

$$\lim_{k\to\infty} \frac{\mathrm{k}}{m+\mathrm{k}} < 1.$$

Such that, $m > 0$ and diverges for

$$\lim_{k\to\infty} \frac{\mathrm{k}}{m+\mathrm{k}} > 1.$$

If m<0, it means that B(m, n) diverges in either scenario (a) or (b) and converges only in case (c).

Suppose $\beta = \frac{n}{m+n}$, then the radius of convergence

$$\frac{1}{\beta} = \frac{m+n}{n} = 1 + \frac{m}{n}.$$

For this, three possible cases will be considered for radius of convergence:

a. The radius of convergence is 2 for m = n.
b. The radius of convergence is larger than two for m>n; and
c. The radius of convergence is less than two when m

Ultimately, in other to determine the radius of convergence, such that 1 < r ≤ μ, where μ ≥ 2. The interval of convergence is then regarded as [-2,2]. For different values m and n, that is, m∗n. Next, we have that

$$T_{m,n}(z) = z + \sum_{k=2}^{\infty}\left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)z^k = z + \sum_{k=2}^{\infty}\left(\frac{(m-1)!\,(n-1)!}{(m+n-1)!}\right)z^k; \quad m,n > 0 \tag{10}$$

is a normalized univalent function with beta function coefficients that is a member of class S.

Hence,

$$T_{m,n} = z + \frac{1}{12}z^2 + \frac{1}{60}z^3 + \frac{1}{280}z^4 + \frac{1}{1260}z^5 + \frac{1}{5544}z^6 + \frac{1}{48048}z^7 + \frac{1}{25740}z^8 + \frac{1}{437580}z^9 + \frac{1}{461890}z^{10} + \cdots \tag{11}$$

But if $m = n$, in equation (10) will yield

$$T_{m,n}(z) = z + \sum_{k=2}^{\infty}\left\{\frac{[\Gamma(n)]^2}{\Gamma(2n)}\right\}z^k = z + \sum_{k=2}^{\infty}\left\{\frac{[(n-1)!]^2}{(2n-1)!}\right\}z^k; \quad n > 0. \tag{12}$$

Many researchers have contributed to the development of beta function, but just for details, see [11, 25].

If a variable X takes on the values 0, 1, 2, 3, 4,... with the corresponding probability, it is considered Poisson distributed.

$e^{-m}, \frac{m^2e^{-m}}{2!}, \frac{m^3e^{-m}}{3!}, \frac{m^4e^{-m}}{4!}, \ldots$ respectively, where m is the parameter. Thus

$$P(X = r) = \frac{m^r e^{-m}}{r!}, \quad r = 0,1,2,3,4,\ldots \tag{13}$$

The series whose coefficients are Poisson distribution probabilities was first proposed by Porwal and Kumar [29].

$$K(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}e^{-m}}{(n-1)!} z^n, \quad z \in \mathbb{E} \tag{14}$$

where $m > 0$. Researchers like Murugusundaramoorthy [13], Frasin and Gharaibeh [5], Gbolagade and Awolere [7], Anwar and Ahmad [1], and others also investigated the function.

Poisson is utilized in software defect control, traffic accident statistics, DNA substitution modeling, overlapping word occurrence distribution modeling, and mostly in businesses that use time for goods and services. It is worthy to note that in 2017, Murugusundaramoothy investigated subclasses of starlike and convex functions involving Poisson distribution series and since then it has been attracting the interest of several other researchers like [7] who also investigated generalized distribution for bi-univalent functions from the perspective of error and Poisson distribution through Bell number, going further in recent time Abo Elyazyd *et* al. [34] investigated Bi-univalent function classes defined by Poisson distribution series to obtain the coefficient bounds $/a_2/$ and $/a_3/$ for subclasses of analytic functions involving subordination principles. However, this author did not investigate the convolution of Poisson distribution with Beta function. It is on this gap this article seeks to address.

Thus, the following are the Hadamard products of the beta function and the Poisson distribution series:

$$K\,\mathrm{T}\,P_\sigma(z,n) = (T_{n,n}K(m,z)) * P_\sigma(z) = z + \sum_{n=2}^{\infty} \frac{(n-1)!\cdot m^{n-1}\cdot e^{-m}}{(2n-1)!}\cdot\frac{a_{n-1}}{G} z^n \qquad (15)$$

In our attempt to use the subordination principles, the Bell number shall be investigated.

The Bell numbers, represented as B_n, indicate how many ways there are to divide a set of n elements into disjoint, non-empty subsets. They are defined recursively by $B_{n+1} = \sum_{k=0}^{n}\binom{n}{k} B_k$

Kumar et al. [9] investigated the generating function:

$$B_n(z) = e^{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots, \qquad (16)$$

Its coefficients are Bell numbers, and it is starlike with regard to unity. Bell numbers are essential to combinatorics and analytic combinatorics, especially when studying set partitions and exponential generating functions [32].

Moreso, the study of **bi-univalency of a generalized distribution series has been considered by numerous researchers in order to examine some interesting properties of the geometric function theory. In this work the convolution of Beta function and Poisson distribution series** shall be investigated along the line of bi-univalent with the aim of establishing new subclasses of normalized analytic functions in the open unit disk E.

In E, a function $t \in H$ is considered bi-univalent if its inverse, t^{-1}, is univalent in E. This concept was first introduced by Lewin [10], who initiated the investigation into coefficient estimates for such functions.

Formally, for $t \in H$, the function t is bi-univalent in $\mathbb{E}$ if there exists an inverse $g = t^{-1}$, also univalent in $\mathbb{E}$, such that:

$$t^{-1}(t(z)) = z, \quad z \in \mathbb{E},$$

$$t(t^{-1}(w)) = w, \quad |w| < r_0(f), \quad \text{with } r_0(f) \geq \frac{1}{4}$$

Where

$$g(w) = t^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \qquad (17)$$

In the past years, a number of contributions have been studied to investigate bi-univalent functions and determine restrictions for initial coefficients. These include Srivastava et al. [31], Frasin & Aouf [6], Murusundamorthy and Yamini [14], Brannan and Taha [35]and Netanyahu [15].

1.1 Lemma and Definition

The following definitions and lemma are essential to our discussion and findings.

Lemma 1.1.1 *(Jahangiri et al.* [8]*):* Let

$$w(z) = w_1z + w_2z^2 + w_3z^3 + w_4z^4 + \cdots \in H \qquad (18)$$

Such that $|w(z)| < 1$ in $\mathbb{E}$. If k is a complex number, then $|w_2 + kw_1^2| \leq \max(1, |k|)$. The inequality is sharp for the function $w(z) = z$ and $w(z) = z^2$.

Lemma 1.1.2: [32] If $p(z) = 1 + c_1z + c_2z^2 + \ldots \in P$ is an analytic function with positive real part and v is a complex variable, then

$$|c_2 - vc_1^2| \leq \begin{cases} 2-4v, & if \quad v \leq 0 \\ 2 & if \quad 0 \leq v \leq 1 \\ 4v-2 & if \quad v \geq 1 \end{cases}$$

For the generalized Starlike and Bounded turning distribution classes, we will determine the initial coefficient bounds $P_oE_G^s(m,\Lambda,b,B_n)$ and $P_oE_G^b(m,\Lambda,b,B_n)$ respectively and they are given as follow:

Definition 1.1.3: For real numbers $0 \leq \Lambda < 1$, $m > 0$, $b \neq 0$, and $B_n(z)$ as defined by (1), then $K\,\mathrm{T}\,P_\sigma \in \phi$ is in the class $P_oE_G^s(m,\Lambda,b,B_n)$ if

$$1 + \frac{1-\Lambda}{b}\left\{\frac{z(K\,\mathrm{T}\,P_\sigma \in \phi(z,n))'}{K\,\mathrm{T}\,P_\sigma \in \phi(z,n)} - 1\right\} \prec B_n(z), \qquad (19)$$

$$1 + \frac{1-\Lambda}{b}\left\{\frac{w(K\,\mathrm{T}\,P_\sigma \in \phi^{-1}(w,n))'}{K\,\mathrm{T}\,P_\sigma \in \phi^{-1}(w,n)} - 1\right\} \prec B_n(w). \qquad (20)$$

Definition 1.1.4: For real numbers $0 \leq \Lambda < 1, m > 0$, $b \neq 0$, and $B_n(z)$ as defined by (1), then $K\,\mathrm{T}\,P_\sigma \in \phi$ is in the class $P_oE_G^b(m,\Lambda,b,B_n)$ if

$$1 + \frac{1-\Lambda}{b}((K\,\mathrm{T}\,P_\sigma \in \phi(z,n))' - 1) \prec B_n(z), \qquad (21)$$

$$1+\frac{1-\Lambda}{b}((K\top P_\sigma^{-1}\in\phi(w,n))'-1)\prec B_n(w). \quad (22)$$

2.0 MAIN RESULTS

Theorem 2.2.1: Consider $b\neq 0$ and

$$B_n(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\cdots.$$

If $K\top P_\sigma$ as defined in (15) as member of the class $P_oE_G^s(m,\Lambda,b,B_n)$, then, we obtain

$$\left|\frac{a_1}{G}\right|\le\max\left\{\frac{3|b|e^m}{(1-\Lambda)m},\frac{6|b|\sqrt{2}e^m}{(1-\Lambda)m},A_1\right\},$$

$$\left|\frac{a_2}{G}\right|\le\max\left\{\left(\frac{6|b|e^m}{(1-\Lambda)m}\right)^2+\frac{30|b|e^m}{(1-\Lambda)m^2},B_1\right\},$$

Where

$$A_1 = \frac{6\sqrt{10}\,|b|e^m}{m\sqrt{|(1-\Lambda)(6be^m-5b-5(1-\Lambda))|}},$$

$$B_1 = \frac{180|b|^2e^{2m}}{m^2|(1-\Lambda)(6be^m-5b-5(1-\Lambda))|}+\frac{30|b|e^m}{(1-\Lambda)^2m^2}.$$

Proof: Let $K\top P_\sigma\in P_oE_G^s(m,\Lambda,b,B_n)$ two Schwarz functions are therefore present. $u,v\in H$ of the form (18) such that

$$1+\frac{1-\Lambda}{b}\left\{\frac{z(K\top P_\sigma(z,n))'}{K\top P_\sigma(z,n)}-1\right\}=B_n(u(z)), \quad (23)$$

$$1+\frac{1-\Lambda}{b}\left\{\frac{w(K\top P_\sigma^{-1}(w,n))'}{K\top P_\sigma^{-1}(w,n)}-1\right\}=B_n(v(w)). \quad (24)$$

By expanding (23) and (24) we obtain the following system

$$(1-\Lambda)C_2\frac{a_1}{G} = bu_1, \quad (25)$$

$$2(1-\Lambda)C_3\frac{a_2}{G}-(1-\Lambda)C_2^2\frac{a_1^2}{G^2} = bu_2+bu_1^2, \quad (26)$$

$$-(1-\Lambda)C_2\frac{a_1}{G} = bv_1, \quad (27)$$

$$-2(1-\Lambda)C_3\frac{a_2}{G}+[4(1-\Lambda)C_3-(1-\Lambda)C_2^2]\frac{a_1^2}{G^2} = bv_2+bv_1^2, \quad (28)$$

Where

$$C_n=\frac{((n-1)!\,m^{n-1}e^{-m}}{(2n-1)!},\quad |C_2|=\frac{me^{-m}}{6},\quad |C_3|=\frac{m^2e^{-m}}{60} \quad (29)$$

It follows from (25) and (27) that

$$u_1=-v_1. \quad (30)$$

and

$$2(1-\Lambda)^2C_2^2\frac{a_1^2}{G^2}=b^2(u_1^2+v_1^2). \quad (31)$$

Also, by (26) and (28), we obtain

$$\{4(1-\Lambda)C_3-2(1-\Lambda)C_2^2\}\frac{a_1^2}{G^2}=b(u_2+v_2)+b(u_1^2+v_1^2). \quad (32)$$

Hence, from (25), (30), (31), and (32), we get

$$\frac{a_1}{G} = \frac{bu_1}{(1-\Lambda)C_2}, \quad (33)$$

$$\frac{a_1^2}{G^2} = \frac{b^2(u_1^2+v_1^2)}{2(1-\Lambda)^2C_2^2}, \quad (34)$$

$$\frac{a_1^2}{G^2} = \frac{b^2(u_2+v_2)}{2[2b(1-\Lambda)C_3-b(1-\Lambda)C_2^2-(1-\Lambda)^2C_2^2]}. \quad (35)$$

Using Lemma 1.1.1; $|u_i|\le 1$ and $|v_i|\le 1$ and (29), it follows from equations (33), (34), and (35) that

$$\left|\frac{a_1}{G}\right| \le \frac{3\,|b|\,e^m}{(1-\Lambda)\,m}, \quad (36)$$

$$\left|\frac{a_1}{G}\right| \le \frac{6\sqrt{2}\,|b|\,e^m}{(1-\Lambda)\,m}, \quad (37)$$

$$\left|\frac{a_1}{G}\right| \le \frac{6\sqrt{10}\,|b|\,e^m}{m\,|(1-\Lambda)(6be^m-5b-5(1-\Lambda))|}. \quad (38)$$

which yield the desired estimates for $\frac{a_1}{G}$

Next, by (26), (28), (30) and (31) we obtain

$$\frac{a_2}{G}=\frac{b^2(u_1^2+v_1^2)}{2(1-\Lambda)^2C_2^2}+\frac{b(u_2-v_2)}{4(1-\Lambda)C_3}. \quad (39)$$

$$\frac{a_2}{G}=\frac{b^2(p_2+q_2)}{2[2b(1-\Lambda)C_3-b(1-\Lambda)C_2^2-(1-\Lambda)^2C_2^2]}+\frac{b(u_2-v_2)}{4(1-\Lambda)C_3}. \quad (40)$$

Once again, since $|u_i|\le 1$ and $|v_i|\le 1$, we infer from equations (39) and (40) that

$$\left|\frac{a_2}{G}\right| \le \left\{\left(\frac{6|b|e^m}{(1-\Lambda)m}\right)^2+\frac{30|b|e^m}{(1-\Lambda)m^2}\right\}, \quad (41)$$

$$\left|\frac{a_2}{G}\right| \le \frac{180|b|^2e^{2m}}{m^2|(1-\Lambda)(6be^m-5b-5(1-\Lambda))|}+\frac{30|b|e^m}{(1-\Lambda)^2m^2}. \quad (42)$$

This completes the proof.

By specializing some parameters, new Corollary can be obtained

Theorem 2.2.2: Consider $b\neq 0$ and

$$B_n(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\cdots.$$

If $K\top P_\sigma$ as defined in (15) as a member of the class $P_oE_G^s(m,\Lambda,b,B_n)$, then we obtain

$$\left|\frac{a_2}{G}-\mu\frac{a_1^2}{G^2}\right|=\begin{cases}\frac{30|b|e^m}{(1-\Lambda)m^2}, & 0\le g(\zeta)\le M\\ 2|g(\zeta)||b|, & g(\zeta)\ge M\end{cases}$$

Where

$$M=\frac{15|b|e^m}{(1-\Lambda)m^2}$$

Proof: From (40) and (35), we have

$$\frac{a_2}{G}-\mu\frac{a_1^2}{G^2}=\frac{(1-\mu)b^2(u_2+v_2)}{2[2b(1-\Lambda)C_3-b(1-\Lambda)C_2^2-(1-\Lambda)^2C_2^2]}+\frac{b(u_2-v_2)}{4(1-\Lambda)C_3}$$

$$=b\left[\left(g(\zeta)+\frac{1}{4(1-\Lambda)C_3}\right)u_2+\left(g(\zeta)-\frac{1}{4(1-\Lambda)C_3}\right)v_2\right]$$

Where

$$g(\zeta)=\frac{(1-\mu)b}{2[2b(1-\Lambda)C_3-b(1-\Lambda)C_2^2-(1-\Lambda)^2C_2^2]}$$

In view of (29), we can establish that

$$\left|\frac{a_2}{G}-\mu\frac{a_1^2}{G^2}\right|=\begin{cases}\frac{30|b|e^m}{(1-\Lambda)m^2}, & 0\le g(\zeta)\le M\\ 2|g(\zeta)||b|, & g(\zeta)\ge M\end{cases}\quad(43)$$

Where

$$M=\frac{15|b|e^m}{(1-\Lambda)m^2}$$

This completes the proof.

By specializing some parameters, new Corollary can be obtained

Theorem 2.2.3: Let $b\neq 0$ and

$$B_n(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\cdots.$$

If KTP_σ as defined in (15) as a member of the class $P_oE_G^b(m,\Lambda,b,B_n)$, then we obtain

$$\left|\frac{a_1}{G}\right|\le\max\left\{\frac{3|b|e^m}{(1-\Lambda)m},\frac{18|b|\sqrt{2}e^m}{(1-\Lambda)m},A_2\right\},$$

$$\left|\frac{a_2}{G}\right|\le\max\left\{\left(\frac{3|b|e^m}{(1-\Lambda)m}\right)^2+\frac{20|b|e^m}{(1-\Lambda)m^2},B_2\right\},$$

Where

$$A_2\&=\frac{6\sqrt{10}\,|b|e^m}{m\sqrt{|(1-\Lambda)(9be^m-20(1-\Lambda))|}},$$

$$B_2\&=\frac{180|b|^2e^{2m}}{m^2|(1-\Lambda)(9be^m-20(1-\Lambda))|}+\frac{20|b|e^m}{(1-\Lambda)m^2}.$$

Proof: Let $KTP_\sigma\in P_oE_G^b(m,\Lambda,b,B_n)$ two Schwarz functions are therefore present $p,q\in H$ of the form (18) such that

$$1+\frac{1-\Lambda}{b}((KTP_\sigma(z,n))'-1)=B_n(p(z)),\quad(44)$$

$$1+\frac{1-\Lambda}{b}((KTP_\sigma^{-1}(w,n))'-1)=B_n(q(w)).\quad(45)$$

From (44) and (45), we deduce that

$$2(1-\Lambda)C_2\frac{a_1}{G}\&=bp_1,\quad(46)$$

$$3(1-\Lambda)C_3\frac{a_2}{G}\&=bp_2+bp_1^2,\quad(47)$$

$$-2(1-\Lambda)C_2\frac{a_1}{G}\&=bq_1,\quad(48)$$

$$-3(1-\Lambda)C_3\frac{a_2}{G}+6(1-\Lambda)C_3\frac{a_1^2}{G^2}\&=bq_2+bq_1^2,\quad(49)$$

Now, from (46) and (48), we obtain

$$p_1=-q_1.\quad(50)$$

and

$$8(1-\Lambda)^2C_2^2\frac{a_1^2}{G^2}=b^2(p_1^2+q_1^2).\quad(51)$$

Also, by (47) and (49), we obtain

$$6(1-\Lambda)C_3\frac{a_1^2}{G^2}=b(p_2+q_2)+b(p_1^2+q_1^2).\quad(52)$$

Hence, from (46), (50), (51), and (52), we derive

$$\frac{a_1}{G}\&=\frac{bp_1}{2(1-\Lambda)C_2},\quad(53)$$

$$\frac{a_1^2}{G^2}\&=\frac{b^2(p_1^2+q_1^2)}{8(1-\Lambda)^2C_2^2},\quad(54)$$

$$\frac{a_1^2}{G^2}\&=\frac{b^2(p_2+q_2)}{2[3b(1-\Lambda)C_3-4(1-\Lambda)^2C_2^2]}.\quad(55)$$

Applying $|p_i|\le 1$ and $|q_i|\le 1$ to (53), (54) and (55) we obtain the desired results

$$\left|\frac{a_1}{G}\right|\&\le\frac{3|b|e^m}{(1-\Lambda)m},\quad(56)$$

$$\left|\frac{a_1}{G}\right|\&\le\frac{18\sqrt{2}\,|b|\,e^m}{(1-\Lambda)\,m},\quad(57)$$

$$\left|\frac{a_1}{G}\right|\&\le\frac{6\sqrt{10}\,|b|\,e^m}{m\,|(1-\Lambda)(9be^m-201-\Lambda))|}.\quad(58)$$

Similarly, by (47), (49), (50) and (51) we derive

$$\frac{a_2}{G}=\frac{b^2(p_1^2+q_1^2)}{8(1-\Lambda)^2C_2^2}+\frac{b(p_2-q_2)}{6(1-\Lambda)C_3}.\quad(59)$$

Again, using (55) we get

$$\frac{a_2}{G}=\frac{b^2(u_2+v_2)}{2[3b(1-\Lambda)C_3-4(1-\Lambda)^2C_2^2]}+\frac{b(p_2-q_2)}{6(1-\Lambda)C_3}.\quad(60)$$

By the same argument of lemma 1.1.1, we infer

$$\left|\frac{a_2}{G}\right|\&\le\left\{\left(\frac{3|b|e^m}{(1-\Lambda)m}\right)^2+\frac{20|b|e^m}{(1-\Lambda)m^2}\right\},\quad(61)$$

$$\left|\frac{a_2}{G}\right|\&\le\frac{180|b|^2e^m}{m^2|(1-\Lambda)(9b-20(1-\Lambda)e^{-m})|}+\frac{20|b|e^m}{(1-\Lambda)m^2}.\quad(62)$$

This completes the proof.

By specializing some parameters, new Corollary can be obtained

Theorem 2.2.4: Let $b\neq 0$ and

$$B_n(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\cdots.$$

If KTP_σ as defined in (15) belongs to the class $P_o E_G^b(m, \Lambda, b, B_n)$, then we obtain

$$\left|\frac{a_2}{G} - \eta \frac{a_1^2}{G^2}\right| = \begin{cases} \frac{20|b|e^m}{(1-\Lambda)m^2}, & 0 \le r(\zeta) \le N \\ 2|r(\zeta)||b|, & r(\zeta) \ge N \end{cases} \quad (63)$$

Where

$$N = \frac{10|b|e^m}{(1-\Lambda)m^2}$$

Proof: From (60) and (55), we have

$$\frac{a_2}{G} - \eta \frac{a_1^2}{G^2} = \frac{(1-\mu)b^2(p_2+q_2)}{2[3b(1-\Lambda)C_3 - 4(1-\Lambda)^2C_2^2]} + \frac{b(p_2-q_2)}{6(1-\Lambda)C_3}$$

$$= b\left[\left(r(\zeta) + \frac{1}{6(1-\Lambda)C_3}\right)p_2 + \left(r(\zeta) - \frac{1}{6(1-\Lambda)C_3}\right)q_2\right]$$

Where

$$r(\zeta) = \frac{(1-\mu)b}{2[3b(1-\Lambda)C_3 - 4(1-\Lambda)^2C_2^2]}$$

In view of (29), we can establish that

$$\left|\frac{a_2}{G} - \eta \frac{a_1^2}{G^2}\right| = \begin{cases} \frac{20|b|e^m}{(1-\Lambda)m^2}, & 0 \le r(\zeta) \le N \\ 2|r(\zeta)||b|, & r(\zeta) \ge N \end{cases} \quad (64)$$

Where

$$N = \frac{10|b|e^m}{(1-\Lambda)m^2}$$

This completes the proof.
By specializing some parameters, new Corollary can be obtained.

Conclusion

This study examines the bi-univalency of the Hardamand product of Poisson distribution series and Beta function insubordination with the Bell numbers to define new subclasses of starlike functions and bounded turning functions which provide an intriguing viewpoint. The coefficient bounds for these subclasses were established by applying Lemmas cite in this work. By carefully varying some parameters new corollaries can be deduced.

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