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Coefficient bounds for subclass of Sigmoid functions involving subordination principle defined by Salagean Differential Operator

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Abstract:

The authors of this research investigated a subclass analytic univalent function from the perspective of sigmoid functions defined by using the Salagean differential operator and subordination principles. Coefficient constraints were found for this subclass and the well-known Fekete Szego inequalities were also mentioned. Therefore, in addition to the standard Fekete-Szegjo problem, the relationship between unified subclasses of analytic univalent functions and a simple logistic activation function to find the initial Taylor series coefficients.

Keywords: **Coefficient bound, Fekete Szego, univalent function, sigmoid functions, Salagean operator.**

1. Introduction

LET A denote the subclass of analytic univalent function and fϵA then we have

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n (z\in U), \tag{1}$$

In terms of geometry, this amounts to reducing or enlarging the domain and perhaps rotating it, but it does not alter the function's univalence. The function f(z) in equation (1) is normalized and meets the set of requirements when it has the form in equation (1) $f(0)=0$, $f'(0)=1$. [1].

The function in (1) are analytic in the open unit disk $U=\{z: z\in \mathbb{C}\ and\ |z|<1\}$, $\mathbb{C}$ being as usual, the set of complex number. Additionally, let S represent the subclass of A functions that are univalent in [2, 3, 4, 5, 6].

The class S* (k) of starlike functions of order k in U and the class K(k) of convex functions of order k in U are two significant and thoroughly studied subclasses of the univalent function class S that follow [7]. Also [6, 8, 9, 10, 11, 21]. We have

$$S^*(k)=\left\{f: f\in S\ and\ \mathcal{R}\left(\frac{zf'(z)}{f(z)}\right)>k \quad (z\epsilon U; 0\le k<1)\right\} \tag{2}$$

and

$$K(k)=\left\{f: f\in S\ and\ \mathcal{R}\left(1+\frac{zf''(z)}{f'(z)}\right)>k \quad (z\epsilon U; 0\le k<1)\right\} \tag{3}$$

It readily follows from the equation (2) and (3) that

$$f(z)\epsilon\ K(k) \Leftrightarrow zf'(z)\epsilon S^*(k) \tag{4}$$

It is well-known that every function $f\epsilon S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \quad (z\ \epsilon\ U)$$

and

$$f(f^{-1}(\omega)=\omega \quad \left(|\omega|<r_0(f): r_0(f)\ge\frac{1}{4}\right).$$

Infact, the inverse function f^{-1} is given by

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots \tag{5}$$

For details, [7, 12].

The analytic functions h which have the series development of the form

$$h(z)=1+c_1z^1+c_2z^2+\ldots=1+\sum_{k=1}^{\infty}c_k z^k, |c_k|\le 2, \quad (k=1,2,3,\ldots) \tag{6}$$

satisfying $h(0)=1$ and $\mathbf{Re}\{h(z)\}>0$ (positive real parts). The well-known Caratheodory function is this one; refer to [13]. An equivalent way to characterize these functions would be as a function subordinate to the well-known Mőbius function, $L_0(z)=\frac{1+z}{1-z}$. In the family of functions of the form like of *h*, the Mőbius function is essential. For such functions, it assumes the extremum in the most extreme problem.

By subordination, it is meant that there exists a function of unit bound ($|\omega(z)|<1$), normalized by $\omega(0)=0$ such that $h(z)=L_0(\omega(z))$. Thus, this gives another representation for h among others as we shall see later. In terms of ω, function h has the form:

$$h(z)=\frac{1+\omega(z)}{1-\omega(z)},\ z\in U. \qquad (7)$$

The unit bound functions, otherwise known as Schwarz functions, have two basic results. For further details [13, 14]
Let the function p be of the form

Notice that p is analytic in U and $P(0)=1$. Also p has positive real part in U and hence $|p_i|\le 2$.
From (8), we have

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2\ldots \qquad (8)$$

Notice that p is analytic in U and $P(0)=1$. Also p has positive real part in U and hence $|p_i|\le 2$.
From (8), we have

$$p(z)(1-u(z))=1+u(z)$$
$$p(z)-p(z)u(z)=1+u(z)$$
$$p(z)-1=u(z)+p(z)u(z) \qquad (9)$$
$$\frac{p(z)-1}{p(z)+1}=u(z).$$

Re-expressing (9) we have

$$(p(z)-1)(p(z)+1)^{-1}=u(z)$$
$$(p_1z+p_2z^2+\ldots)(2+p_1z+p_2z^2+\ldots)^{-1}=u(z)$$
$$2^{-1}\left(p_1z+p_2z^2+p_3z^3+p_4z^4+\ldots\right)\left(1+\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right)^{-1}=u(z)$$

$$u(z)=\left(\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4\right)\begin{pmatrix}1-\left[\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right]+\\ \left[\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right]^2-\\ \left[\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right]^3+\\ \left[\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right]^4-\ldots\end{pmatrix}$$

$$u(z)=\left(\frac{p_1}{2}z+\frac{p_2}{2}z^2+\frac{p_3}{2}z^3+\frac{p_4}{2}z^4+\ldots\right)\begin{pmatrix}1-\frac{p_1}{2}z+\left(\frac{p_1^2}{4}-\frac{p_2}{2}\right)z^2+\\ \left(\frac{p_1p_2}{2}-\frac{p_3}{2}-\frac{p_1^3}{8}\right)z^3+\\ \left(\frac{p_1p_3}{2}+\frac{p_2^2}{4}+\frac{p_1^4}{16}-\frac{3p_1^2p_2}{8}-\frac{p_4}{2}\right)z^4+\ldots\end{pmatrix}$$

$$u(z)=\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}+\frac{p_1}{4}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{4}\right)z^3+\left(\frac{p_4}{2}+\frac{p_1^2}{8}\left(p_2-\frac{p_1^2}{2}\right)+\frac{p_1}{4}\left(p_1p_2-p_3\right)-\frac{p_2^2}{4}\right)z^4+\dots$$

$$u(z)=\frac{1}{2}\left(\begin{array}{l}p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1}{2}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{2}\right)z^3+\\ \left(p_4+\frac{p_1^2}{4}\left(p_2-\frac{p_1^2}{2}\right)+\frac{p_1}{2}\left(p_1p_2-p_3\right)-\frac{p_2^2}{2}\right)z^4+\dots\end{array}\right) \tag{10}$$

The Hadamard product (or convolution) of f(z) given by (1) and the function $\varphi(z)$ left(z\right) which is of the form

$$\varphi(z)=z+\sum_{k=2}^{\infty}\varphi_k z^k$$

Is defined by

$$(f*\varphi)(z)=z+\sum_{k=2}^{\infty}a_k\varphi_k z^k=(\varphi*f)(z)$$

Therefore, $D^n(f*\varphi)(z)=D\big(D^{n-1}(f*\varphi)(z)\big)=z+\sum_{k=2}^{\infty}k^n a_k\varphi_k z^k$ where D^n is defined as the famous Salagean derivative operator [19].

$$D^0f(z)=f(z),D^1f(z)=D\big(f(z)\big)=zf'(z),\cdots$$

$$D^nf(z)=D\big(D^{n-1}f(z)\big)=z+\sum_{k=2}^{\infty}k^n a_k z^k$$

The Hadamard product was also used by [15] to talk about a new class of functions represented by $M_{g,h}(\phi)$.

Due to their engagement in mathematical computations, scientists and engineers who regularly employ special functions place a great deal of importance on the theory of special functions, according to [16]. Although it lacks a precise definition, its applications are found in fields such as computer science and physics. Other disciplines like real analysis, functional analysis, differential equation, algebra, and topology have recently eclipsed the theory of special functions [16]. There are several special functions, but we'll focus on the activation function, also referred to as the sigmoid function or the simple logistic function. We used the term "activation function" to refer to an information-based process that is comparable to how the brain and other biological nerve systems process information. This is made up of several closely connected processing elements, or neurons, that cooperate to process or accomplish a particular task. It cannot be trained to perform a specific activity; instead, it learns by example. The gradient descendent learning procedure of the sigmoid function (simple logistic activation function) can be evaluated in a number of ways, including by truncated series expansion.

The following is the simple logistic activation function:

$$\ell(z)=\frac{1}{1+e^{-z}},z\geq 0 \tag{11}$$

It transfers a relatively broad input domain to a tiny range of outputs, is differentiable, produces real numbers between 0 and 1, rises monotonically, and never loses information because it is a one-to-one function. The sigmoid function is a very useful tool in geometric function theory, as is seen from the above. For details, [5, 17, 18, 21]

Lemma 1.1.1: [12, 14] If a function $p\in P$ is given by

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\dots(z\in U)$$

then $|p_k|\leq 2,\ k\in N$ where P is the family of all functions analytic in U for which $p(0)=1$ and $\operatorname{Re}(p(z))>0,(z\in U)$.

Lamma 1.1.2: [17]: Let ℓ be a sigmoid function and

$$\Phi_{k,m}=2\ell(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left[\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right]^k$$

then $\Phi_{k,m}\in P,|z|<1$ where $\Phi_{k,m}$ is a modified sigmoid function.

Lemma 1.1.3:.[17] Let

$$\Phi_{k,m}(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left[\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right]^k$$

then

$$|\Phi_{k,m}(z)|<2$$

Lemma 1.1.4: [17] If $\Phi_{k,m}\in P$ is starlike then f is a normalized univalent function of the form.

Taking $k=1$, Fadipe – Joseph et al [5] proved that

Lemma 1.1.5: [12]. **If** $p(z)=1+c_1z+c_2z^2+\dots\in \mathbf{P}$ is an analytic function with positive real part and v is a complex variable, then $|c_2-vc_1^2\leq 2\max\{1;|2v-1|\}$.

Lemma 1.1.6: [8] If $p(z)=1+c_1z+c_2z^2+\dots\in P$ is an analytic function with positive real part and v is a complex variable, then

$$|c_2-vc_1^2|\leq\begin{cases}2-4v, & if\quad v\leq 0\\ 2 & if\quad 0\leq v\leq 1\\ 4v-2 & if\quad v\geq 1\end{cases}$$

Remark. Let

$$\Phi(z)=1+\sum_{m=1}^{\infty}C_m z^m$$

Where $C_m = \frac{(-1)(-1)^m}{2m!} |C_m| \le 2,\ m = 1,2,3,\cdots$ this result is sharp for each m [5].

Oladipo [16] studied the class $M_{n,h}^{\alpha,g}(b,\Phi_{k,m})$ of functions $f(z) \in S$ and which satisfies the following condition:

Definition 1. For $b \in C$, Let the class $M_{n,h}^{\alpha,g}(b,\Phi_{k,m})$ denote the subclass of A consisting of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $h(z) = z + \sum_{k=2}^{\infty} h_k z^k$, $g_k > 0, h_k > 0, g_k - h_k > 0$ Satisfying the following conditions:

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} - 1\right] \prec \Phi_{k,m}(z)$$

Where $\alpha \ge 0$, $n \in N_0$, $\Phi_{k,m}$ is a simple logistic sigmoid activation function and D^n is the Salagean derivative operator [19, 20].

In the series form, the sigmoid function looks like this:

$$\ell(z) = \frac{1}{1+e^{-z}} = 1 - e^{-z} + (e^{-z})^3 + (e^{-z})^4 \ \ldots$$

$$\ell(z) = 1 - \left[1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^4}{24} - \ldots\right] + \left[1 - z + \frac{2^2}{2} - \frac{z^3}{6} + \frac{z^4}{24}\ldots\right]^2$$
$$-\left[1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^4}{24} - \ldots\right]^3 + \left[1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^4}{24}\ldots\right]^4 - \ldots$$

$$\ell(z) = 1 - \left[1 - 2z + 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 - \ldots\right] - \left[1 - 3z + \frac{9}{2}z^2 - \frac{9}{2}z^3 + \frac{27}{8}z^4 - \ldots\right] +$$
$$\left[1 - 4z + 8z^2 - \frac{32}{3}z^3 + \frac{32}{3}z^4 - \ldots\right]$$

$$\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots \tag{12}$$

Using (10) in (12) form the composition of function that follows as

$$\ell(\phi_1(z)) = 1 - 2\left[\begin{matrix}\frac{P_1^2}{2} + \left(\frac{P_2}{2} - \frac{P_1^2}{4}\right)z^2 + \left(\frac{P_3}{2} + \frac{P_1}{4}\left(\frac{P_1^2}{2} - P_2\right) - \frac{P_1P_2}{4}\right)z^3 + \\ \left(\frac{P_4}{2} + \frac{P_1}{4}\left(\frac{3P_1P_3}{4} - P_3 - \frac{P_1^3}{8}\right) - \frac{P_2^2}{4}\right)z_4^4 + \ldots\end{matrix}\right] +$$
$$5\left[\frac{P_1}{2}z + \left(\frac{P_2}{2} - \frac{P_1^2}{4}\right)z^2 + \left(\frac{P_3}{2} + \frac{P_1}{4}\left(\frac{P_1^2}{2} - P_2\right) - \frac{P_1P_2}{4}\right)z^3 + \left(\frac{P_4}{2} + \frac{P_1}{4}\left(\frac{3P_1P_3}{4} - P_3 - \frac{P_1^3}{8}\right) - \frac{P_2^2}{4}\right)z^4 + \ldots\right]^2$$
$$-\frac{22}{3}\left[\frac{P_1}{2}z + \left(\frac{P_2}{2} - \frac{P_1^2}{4}\right)z^2 + \left(\frac{P_3}{2} + \frac{P_1}{4}\left(\frac{P_1^2}{2} - P_2\right) - \frac{P_1P_2}{4}\right)z^3 + \left(\frac{P_4}{2} + \frac{P_1}{4}\left(\frac{3P_1P_3}{4} - P_3 - \frac{P_1^3}{8}\right) - \frac{P_2^2}{4}\right)z^4 + ..\right]^3$$
$$+\frac{95}{12}\left[\frac{P_1}{2}z + \left(\frac{P_2}{2} - \frac{P_1^2}{4}\right)z^2 + \left(\frac{P_3}{2} + \frac{P_1}{4}\left(\frac{P_1^2}{2} - P_2\right) - \frac{P_1P_2}{4}\right)z^3 + \left(\frac{P_4}{2} + \frac{P_1}{4}\left(\frac{3P_1P_3}{4} - P_3 - \frac{P_1^3}{8}\right) - \frac{P_2^2}{4}\right)z^4 + ..\right]^4$$

$$\ell(\phi_1(z)) = 1 - P_1 z - \left(P_2 - \frac{7}{4}P_1^{\,2}\right)z^2 - \left(P_3 + \frac{P_1}{2}\left(\frac{29}{6}P_1^{\,2} - 7P_2\right)\right)z^3$$
$$-\left(P_4 + \frac{P_1}{2}\left(\frac{3P_1P_3}{4} + 5P_1P_2 - 6P_3 - \frac{815}{96}P_1^{\,3}\right) + \frac{P_2}{4}(27 - 7P_2)\right)z^4 + \qquad (13)$$

Definition 2: For $b \in C$, Let the class $W_{n,h}^{\alpha,g}(b,\phi)$ denote the subclass of S consisting of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $h(z) = z + \sum_{k=2}^{\infty} h_k z^k$,

$g_k > 0, h_k > 0, g_k - h_k > 0$

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{\left(D^n(f*g)(z)\right)'}{\left(D^n(f*h)(z)\right)'} - 1\right] \prec \ell(\varphi(z)) \qquad (14)$$

$$\ell(\phi_1(z)) = 1 - P_1 z - \left(P_2 - \frac{7}{4}P_1^{\,2}\right)z^2 - \left(P_3 + \frac{P_1}{2}\left(\frac{29}{6}P_1^{\,2} - 7P_2\right)\right)z^3$$
$$-\left(P_4 + \frac{P_1}{2}\left(\frac{3P_1P_3}{4} + 5P_1P_2 - 6P_3 - \frac{815}{96}P_1^{\,3}\right) + \frac{P_2}{4}(27 - 7P_2)\right)z^4 +$$

where b is a non-zero complex number, $\alpha \geq 0, n \in N_0$, ℓ is a function of sigmoid and D^n is the Salagean derivative operator [19, 20].

The following is based on the Hadamard product principle:

$D^n(f*g)(z) = z + \sum_{k=2}^{\infty} k^n a_k\, g_k z^k$ (15)

$D^n(f*h)(z) = z + \sum_{k=2}^{\infty} k^n a_k\, h_k z^k$ (16)

$(D^n(f*g)(z))' = 1 + \sum_{k=2}^{\infty} k^{n+1} a_k\, g_k z^{k-1}$ (17)

$(D^n(f*h)(z))' = 1 + \sum_{k=2}^{\infty} k^{n+1} a_k\, h_k z^{k-1}$ (18)

Using (3.11) and (3.12) in linear fractional form we have:

$$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} = \frac{1 + \sum_{k=2}^{\infty} k^{n+1} a_k\, g_k z^{k-1}}{1 + \sum_{k=2}^{\infty} k^{n+1} a_k\, h_k z^{k-1}}$$

$$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} = \frac{1 + 2^{n+1}a_2g_2z + 3^{n+1}a_3g_3z^2 + 4^{n+1}a_4g_4z^3 + 5^{n+1}a_5g_5z^4 + \cdots}{1 + 2^{n+1}a_2h_2z + 3^{n+1}a_3h_3z^2 + 4^{n+1}a_4h_4z^3 + 5^{n+1}a_5h_5z^4 + \cdots}$$

$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} = (1 + 2^{n+1}a_2g_2z + 3^{n+1}a_3g_3z^2 + 4^{n+1}a_4g_4z^3 + 5^{n+1}a_5g_5z^4 + \cdots)(1 + 2^{n+1}a_2h_2z + 3^{n+1}a_3h_3z^2 + 4^{n+1}a_4h_4z^3 + 5^{n+1}a_5h_5z^4 + \cdots)^{-1}$

$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} = (1 + 2^{n+1}a_2g_2z + 3^{n+1}a_3g_3z^2 + 4^{n+1}a_4g_4z^3 + 5^{n+1}a_5g_5z^4 + \cdots)(1 - [2^{n+1}a_2h_2z + 3^{n+1}a_3h_3z^2 + 4^{n+1}a_4h_4z^3 + 5^{n+1}a_5h_5z^4 + \cdots] + [2^{n+1}a_2h_2z + 3^{n+1}a_3h_3z^2 + 4^{n+1}a_4h_4z^3 +$

$5^{n+1}a_5h_5z^4+\cdots]^2+[2^{n+1}a_2h_2z+3^{n+1}a_3h_3z^2+4^{n+1}a_4h_4z^3+5^{n+1}a_5h_5z^4+\cdots]^3+[2^{n+1}a_2h_2z+3^{n+1}a_3h_3z^2+4^{n+1}a_4h_4z^3+5^{n+1}a_5h_5z^4+\cdots]^4)$

$$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'}=(1+2^{n+1}a_2g_2z+3^{n+1}a_3g_3z^2+4^{n+1}a_4g_4z^3+5^{n+1}a_5g_5z^4+\cdots)(1-2^{n+1}a_2h_2z+(2^{n+1}a_2h_2-3^{n+1}a_3h_3)z^2+(2.6^{n+1}a_2a_3h_2h_3-2^{n+3}a_2^3h_2^3-4^{n+1}a_4h_4)z^3+(9^{n+1}a_3^2h_3^2+2.8^{n+1}a_2h_2a_4h_4+2^{2n+4}a_2^4h_2^4-2.6^{n+1}a_2^2h_2^2a_3h_3(1+2^{n+1})-5^{n+1}a_5h_5)z^4+\cdots)$$

$$\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'}=1+2^{n+1}a_2(g_2-h_2)z+\left(2^{n+2}a_2^2(h_2^2-2^ng_2h_2)+3^{n+1}a_3(g_3-h_3)\right)z^2+\left(4^{n+1}a_4(g_4-h_4)+2^{n+3}a_2^3h_2^2(2^ng_2-h_2)+6^{n+1}a_2a_3h_3(g_3-g_2+2a_3h_2)\right)z^3+(5^{n+1}a_5g_5+2.6^{n+1}a_3a_2^2h_2(g_3h_2+2^{n+1}g_2h_3)-2^{2n}(2^{n+3}a_4a_2g_4h_2+2^{n+3}h_4+2^4a_2^4g_2h_2^3)-9^{n+1}a_3^2g_3h_3)z^4+\cdots \tag{19}$$

Also using (15) and (16)

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=\frac{z+\sum_{k=2}^{\infty}k^na_kg_kz^k}{z+\sum_{k=2}^{\infty}k^na_kh_kz^k}=\frac{z+2^na_2g_2z^2+3^na_3g_3z^3+4^na_4g_4z^4+\cdots}{z+2^na_2h_2z^2+3^na_3h_3z^3+4^na_4h_4z^4+\cdots}$$

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=(z+2^na_2g_2z^2+3^na_3g_3z^3+4^na_4g_4z^4+\cdots)(z+2^na_2h_2z^2+3^na_3h_3z^3+4^na_4h_4z^4+\cdots)^{-1}$$

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=(1+2^na_2g_2z+3^na_3g_3z^2+4^na_4g_4z^3+\cdots)(1+2^na_2h_2z+3^na_3h_3z^2+4^na_4h_4z^3+\cdots)^{-1}$$

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=(1+2^na_2g_2z+3^na_3g_3z^2+4^na_4g_4z^3+\cdots)(1-[2^na_2h_2z+3^na_3h_3z^2+4^na_4h_4z^3+\cdots]+[2^na_2h_2z+3^na_3h_3z^2+4^na_4h_4z^3+\cdots]^2-[2^na_2h_2z+3^na_3h_3z^2+4^na_4h_4z^3+\cdots]^3+[2^na_2h_2z+3^na_3h_3z^2+4^na_4h_4z^3+\cdots]^4-)$$

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=(1+2^na_2g_2z+3^na_3g_3z^2+4^na_4g_4z^3+\cdots)(1-2^na_2h_2z+(4^na_2^2h_2^2-3^na_3h_3)z^2+(8^na_2^3h_2^3+2.6^na_2h_2a_3h_3-4^na_4h_4)z^3+(16^na_2^4h_2^4+3.12^na_2^2h_2^2a_3h_3+2.8^na_2h_2a_4h_4+3^{2n}a_3^2h_3^2-5^na_5h_5)z^4+\cdots)$$

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)}=1+2^na_2(g_2-h_2)z+\left(3^na_3(g_3-h_3)-2^{2n}a_2^2h_2(g_2-h_2)\right)z^2+\left(4^na_4(g_4-h_4)+6^na_2a_3h_2(2h_3-g_3-g_2)\right)z^3+\left(16^na_2^4h_2^3(g_2-h_2)+12^na_2^2a_3(3h_2^2h_3+2g_2h_3+g_3h_2^2)+8^na_2a_4(2.h_2h_4-g_2h_4-g_4h_2)+3^{2n}a_2^2h_3(g_3-h_3)+5^na_5(g_5-h_5)\right)z^4+\cdots \tag{20}$$

$$1+\frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)}+\alpha\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'}-1\right]=1+\frac{2^n a_2(g_2-h_2)(1+\alpha)}{b}z+$$
$$\frac{\left(3^n a_3(g_3-h_3)(1+2\alpha)-2^n a_2^2 h_2\left(2^n g_2(3\alpha+1)+h_2\left(2^n\alpha-4\alpha-2^n\right)\right)\right)z^2}{b}+$$
$$\frac{\left(4^n a_4(g_4-h_4)(1+3\alpha)-2.6^n a_2a_3h_2h_3\left(6(g_3-h_2)+3\left(g_2^2-g_3^2\right)+2h_2h_3-h_2a_2(g_3+h_3)\right)(1-\alpha)\alpha\right)z^3}{b}+$$
$$+\left(\frac{\begin{array}{l}\left[\left[\begin{array}{l}16^n a_2^4h_2^3(g_2-h_2)+12^n a_2^2a_3\left(3h_2^2h_3+2g_2h_3+g_3h_2^2\right)\\+8^n a_2a_4(2h_2h_4-g_2h_4-g_4h_2)+3^{2n}a_2^2h_3(g_3-h_3)\end{array}\right]\right]\\(1-\alpha)+\left[\begin{array}{l}2.6^{n+1}a_3a_2^2h_2\left(g_3h_2+2^{n+1}g_2h_3\right)-2^{2n}\left(2^{n+3}a_4a_2g_4h_2+2^{n+3}h_4+2^4a_2^4g_2h_2^3\right)\\-9^{n+1}a_3^2g_3h_3\end{array}\right]\alpha\\+5^n a_5(g_5-h_5-\alpha(g_5-h_5-5))\end{array}}{b}\right)z^4+\dots$$

(21)

2.0 MAIN RESULTS

Theorem 2.1.1

Let $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-\dots$ where $\ell\in A$ is an activation function for a logistic sigmoid and $\ell'(0)>0$.

If $f(z)$ is given by eqn 1 belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$, where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,\ \alpha\geq 0,\ n\in N_0,\ k\geq 2$ and D^n is the derivative operator for Salagean, then

$$|a_2|\leq\frac{b}{2^{n-1}(g_2-h_2)(1+\alpha)} \tag{22}$$

$$|a_3|\leq\frac{8.2^n b(g_2-h_2)^2(1+\alpha)^2-(2^n.4.7.(g_2-h_2)^2(1+\alpha)^2b+16b^2h_2(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)))}{3^n.4.2^n(g_3-h_3)(g_2-h_2)^2(1+\alpha)^2(1+2\alpha)} \tag{23}$$

$$|a_4|\leq\frac{22b}{4^n.3(g_4-h_4)(1+3\alpha)}+2.6^n\times h_2h_3\left(\frac{2^n\times 10b^2(g_2-h_2)^2(1+\alpha)^2+8b^3h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{3^n2^{4n}(g_4-h_4)(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\right)\times$$
$$\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3-\frac{2h_2b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha \tag{24}$$

Proof: If $f\in \mathbf{W}_{n,h}^{\alpha,g}(b,\ell)$, then there exist a Schwarz function $u(z)$ with conditions that conditions that $|u(z)|<1$ and $u(0)=0$ such that

$$1+\frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)}+\alpha\frac{\left[D^n(f*g)(z)\right]'}{\left[D^n(f*h)(z)\right]'}-1\right]=\ell(\phi(z)).$$

(25)

Calculations reveal that

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)} =$$

$$1+2^n a_2(g_2-h_2)z+(3^n a_3(g_3-h_3)-2^{2n}a_2^2h_2(g_2-h_2))z^2+(4^n a_4(g_4-h_4)+ \quad (26)$$
$$6^n a_2a_3h_2(2h_3-g_3-g_2))z^3+(16^n a_2^4h_2^3(g_2+h_2)+12^n a_2^2a_3(3h_2^2h_3+2g_2h_3+g_3h_2^2)$$
$$+8^n a\ a_4(2h_3h_4-g_2h_4-g_4h_2)+3^{2n}a_2^2h_3(g_3-h_3)+5^n a_5(g_5-h_5))z^4+\ldots$$

$$\frac{\left[D^n(f*g)(z)\right]'}{\left[D^n(f*h)(z)\right]'}=1+2^{n+1}a_2(g_2-h_2)z+(2^{n+2}a_2^2(h_2^2-2^n g_2h_2)+3^{n+1}a_3(g_3-h_3))z^2+$$
$$(4^{n+1}a_4(g_4-h_4)+2^{n+3}a_2^3h_2^2(2^n g_2h_2)+6^{n+1}a_2a_3h_3(g_3-g_2+2a_2h_2))z^3+ \quad (27)$$
$$(5^{n+1}a_5g_5+2.6^{n+1}a_3a_2^2h_2(g_3h_2+2^{n+1}g_2h_3)-2^{2n}(2^{n+3}a_4a_2g_4h_2+2^{n+3}h_4+2^4a_2^4g_2h_2^3)$$
$$-9^{n+1}a_3^2g_3h_3)z^4+\ldots$$

and the manifestation of the Taylor series of $\ell(\phi(z))$ is given as

$$\ell(\phi(z))=1-p_1z-(p_2-\frac{7}{4}p_1^2)z^2-(p_3+\frac{p_1}{2}(\frac{29}{6}p_1^2-7p_2))z^3-$$
$$(p_4+\frac{p_1}{2}(\frac{3}{4}p_1p_3+5p_1p_2-6p_3-\frac{815}{96}p_1^3)+\frac{p_2}{4}(27-7p_2))z^4+\ldots \quad (28)$$

From (25), (26), (27) and (28) we have

$$2^n a_2(g_2-h_2)(1+\alpha)=-p_1b \quad (29)$$

$$3^n a_3(g_3-h_3)(1+2\alpha)=\left[-p_2+\frac{7}{4}p_1^2+2^n a_2^2h_2(2^n g_2(3\alpha+1)+h_2(2^n(\alpha-1)-4\alpha))\right]b \quad (30)$$

$$\left(3^n a_3(g_3-h_3)(1+2\alpha)-2^n a_2{}^2h_2\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)\right)=-bp_2+\frac{7}{4}bp_1{}^2$$

$$3^n a_3(g_3-h_3)(1+2\alpha)-\frac{2^n p_1{}^2b^2h_2}{2^{2n}(g_2-h_2)^2(1+\alpha)^2}\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)=-bp_2+\frac{7}{4}bp_1{}^2$$

$$3^n a_3(g_3-h_3)(1+2\alpha)=-bp_2+\frac{7}{4}bp_1{}^2+\frac{2^{-n}p_1{}^2b^2h_2}{(g_2-h_2)^2(1+\alpha)^2}\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)$$

$$a_3=\frac{1}{3^n(g_3-h_3)(1+2\alpha)}\left[-bp_2+\frac{7}{4}bp_1{}^2+\frac{2^{-n}p_1{}^2b^2h_2}{(g_2-h_2)^2(1+\alpha)^2}\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)\right]$$

$$a_3=\frac{1}{3^n(g_3-h_3)(1+2\alpha)}\left[\left(\frac{7p_1{}^2-4p_2}{4}\right)b+\frac{p_1{}^2b^2h_2\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{2^n(g_2-h_2)^2(1+\alpha)^2}\right]$$

Using $|p_i|\le 2$, we have

$$|a_3|\le \frac{2^n.8.b(g_2-h_2)^2(1+\alpha)^2-(2^n.4.7(g_2-h_2)^2(1+\alpha)^2b+16b^2h_2\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{4.3^n.2^n(g_3-h_3)(g_2-h_2)^2(1+\alpha)^2(1+2\alpha)}$$

$$4^n a_4(g_4-h_4)(1+3\alpha)-2.6^n a_2a_3h_2h_3(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3-h_2a_2(g_3+h_3)(1-\alpha)\alpha)=-bp_3-\frac{p_1}{2}\left(\frac{29}{6}p_1{}^2-7p_2\right)b \quad (31)$$

$$4^n a_4(g_4-h_4)(1+3\alpha)-2.6^n\left(\frac{p_1b}{2^n(g_2-h_2)(1+\alpha)}\right)\left(\frac{(7p_1{}^2-4p_2)b}{3^n4(g_3-h_3)(1+2\alpha)}+\frac{p_1{}^2b^2h_2\left(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{3^n2^n(g_3-h_3)(g_2-h_2)^2(1+\alpha)^2(1+2\alpha)}\right)\times h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{h_2p_1b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha=-bp_3-\frac{p_1}{2}\left(\frac{29}{6}p_1{}^2-7p_2\right)b$$

$$4^n a_4(g_4-h_4)(1+3\alpha) = -bp_3 - \frac{p_1}{2}\left(\frac{29}{6}p_1{}^2 - 7p_2\right)b - 2.6^n\left(\frac{(7p_1{}^3-4p_1p_2)b^2}{2^n.3^n4(g_3-h_3)(g_2-h_2)(1+2\alpha)(1+\alpha)} + \frac{p_1{}^3b^3h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{3^n2^n(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)}\right)h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{h_2p_1b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha$$

$$a_4 = \frac{1}{4^n(g_4-h_4)(1+3\alpha)}\left(-bp_3 - \frac{p_1}{2}\left(\frac{29}{6}p_1{}^2 - 7p_2\right)b\ - 2.6^n\left(\frac{(7p_1{}^3-4p_1p_2)b^2}{2^n.3^n4(g_3-h_3)(g_2-h_2)(1+2\alpha)(1+\alpha)} + \frac{p_1{}^3b^2h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{3^n2^n(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)}\right)h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{h_2p_1b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha\right)$$

Using $p_i = 2$

$$a_4 = \frac{1}{4^n(g_4-h_4)(1+3\alpha)}\left(-2b - \frac{16b}{3} - 2.6^n\left(\frac{10b^2}{2^n.3^n4(g_3-h_3)(g_2-h_2)(1+2\alpha)(1+\alpha)} - \frac{8b^3h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{3^n2^n(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)}\right)h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{2h_2b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha\right)$$

Further simplification gives

$$a_4 = \frac{1}{4^n(g_4-h_4)(1+3\alpha)}\left(\frac{-22b}{3} - 2.6^n\left(\frac{2^n\times 10b^2(g_2-h_2)^2(1+\alpha)^2+8b^3h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{4.3^n.2^n(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)}\right)h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{2h_2b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha\right)$$

Now using $|p_i| \le 2$, we have

$$|a_4| \le \frac{1}{4^n(g_4-h_4)(1+3\alpha)}\left(\frac{22b}{3} + 2.6^n\left(\frac{2^n\times 10b^2(g_2-h_2)^2(1+\alpha)^2+8b^3h_2\left(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)\right)}{4.3^n.2^n(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)}\right)h_2h_3\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3+\frac{2h_2b(g_3+h_3)}{2^n(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha\right)$$

From (29), (30) and (31) the desire results follow.

Corollary 2.1.2: when $n = 0$

Given that $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ where $\ell \in A$ is an activation function for a logistic sigmoid and $\ell'(0) > 0$. If $f(z)$ is given by eqn 1 belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$, where b is a non-zero complex number, $g_k > 0,\ h_k > 0,\ g_k - h_k > 0,\ \alpha \ge 0,\ n \in N_0,\ k \ge 2$ and D^n is the Salagean derivative operator then

$$|a_2| \le \frac{2b}{(g_2-h_2)(1+\alpha)}$$

$$|a_3| \le \frac{8b(g_2-h_2)^2(1+\alpha)^2 - (4.7.(g_2-h_2)^2(1+\alpha)^2b + 16b^2h_2(g_2(3\alpha+1)-h_2(3\alpha+1)))}{4(g_3-h_3)(g_2-h_2)^2(1+\alpha)^2(1+2\alpha)}$$

$$|a_4| \le \frac{22b}{3(g_4-h_4)(1+3\alpha)} + 2h_2h_3\left(\frac{10b^2(g_2-h_2)^2(1+\alpha)^2+8b^3h_2\left(g_2(3\alpha+1)-h_2(3\alpha+1)\right)}{(g_4-h_4)(g_3-h_3)(g_2-h_2)^3(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\right)\times\left(6(g_3-h_2)+3(g_2{}^2-g_3{}^2)+2h_2h_3-\frac{2h_2b(g_3+h_3)}{(g_2-h_2)(1+\alpha)}\right)(1-\alpha)\alpha$$

Corollary 2.1.3: when $\alpha = 0$,

Given that $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ where $\ell \in A$ is an activation function for a logistic sigmoid and $\ell'(0) > 0$. If $f(z)$ is given by eqn 1 belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$, where b is a non-zero complex number, $g_k > 0,\ h_k > 0,\ g_k - h_k > 0,\ \alpha \ge 0,\ n \in N_0,\ k \ge 2$ and D^n is the Salagean derivative operator then

$$|a_2| \le \frac{b}{2^{n-1}(g_2-h_2)}$$

$$|a_3| \le \frac{8.2^nb(g_2-h_2)^2 - \left(2^n.4.7(g_2-h_2)^2\right)b + 16b^2h_2.2^n(g_2-h_2))}{3^n.4.2^n(g_3-h_3)(g_2-h_2)^2}$$

$$|a_4| \le \frac{22b}{4^n.3(g_4-h_4)}$$

Corollary 2.1.4: when $\alpha = 0, n = 0$

Given that $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ where $\ell \in A$ is an activation function for a logistic sigmoid and $\ell'(0) > 0$. If $f(z)$ is given by eqn 1 belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$, where b is a non-zero complex number, $g_k > 0$, $h_k > 0$, $g_k - h_k > 0$, $\alpha \geq 0$, $n \in N_0$, $k \geq 2$ and D^n is the Salagean derivative operator then

A. $|a_2| \leq \frac{2b}{(g_2-h_2)}$

$$|a_3| \leq \frac{8b(g_2-h_2)-(28(g_2-h_2)b+16b^2h_2)}{4(g_3-h_3)(g_2-h_2)}$$

$$|a_4| \leq \frac{22b}{3(g_4-h_4)}$$

The Corollaries 2.1.2, 2.1.3 and 2.1.4 are new results. They increased monotonically. The coefficient bound $|a_2|$ obtained here is a reciprocal to result in [16] owing to the use of the fact that Sigmoid function was used as a composite function involving subordination principle. This suggests that the new coefficient bound $|a_2|$ becomes upper bound to the one provided by [16]. The use of sigmoid function in subordination principle suggests a higher degree of information management strategy.

2.2 Fekete-Szego inequalities for class $W_{n,h}^{\alpha,g}(b,\ell)$

Theorem 2.2.1

Let $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k > 0$, $h_k > 0$, $g_k - h_k > 0$, $k \geq 2$,

$$\left|a_3 - \mu a_2^2\right| \leq 2k \begin{cases} 1 - \dfrac{7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 + 4b^2h_2\left(2^n g_2(3\alpha+1) + h_2\left(2^n\alpha - 4\alpha - 2^n\right)\right) - \mu b^2 4.3^n(1+2\alpha)(g_3-h_3)}{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2} & \text{, if } \mu \leq K_1 \\ 1 & \text{, if } K_1 \leq \mu \leq K_2 \\ \dfrac{7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 + 4b^2h_2\left(2^n g_2(3\alpha+1) + h_2\left(2^n\alpha - 4\alpha - 2^n\right)\right) - \mu b^2 4.3^n(1+2\alpha)(g_3-h_3)}{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2} - 1 & \text{, if } \mu \geq K_2 \end{cases} \quad (32)$$

where

$$K = \frac{b}{3^n(1+2\alpha)(g_3-h_3)}$$

$$K_1 = \frac{7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 - 2^2b^2h_2\left(2^n g_2(3\alpha+1) + h_2\left(2^n\alpha - 4\alpha - 2^n\right)\right)}{2^2.3^n b^2(1+2\alpha)(g_3-h_3)},$$

$$K_2 = \frac{4.2^{2n}(g_2-h_2)^2(1+\alpha)^2 + 2^{2n}.7(g_2-h_2)^2(1+\alpha)^2 - 4h_2b^2(2^n g_2(3\alpha+1) + h_2(2^n\alpha - 4\alpha - 2^n))}{4.3^n(g_3-h_3)(1+2\alpha)b^2}.$$

Proof: Using the coefficient bounds from Theorem 3.1, this gives:

$$a_2 = -\frac{-bp_1}{2^n(g_2-h_2)(1+\alpha)}$$

and

$$a_3 = \frac{-p_2}{3^n(g_3-h_3)(1+2\alpha)} + \frac{7p_1^2 b}{3^n.4(g_3-h_3)(1+2\alpha)} + \frac{b^2 P_1^2 h_2\left(2^n g_2(3\alpha+1)+h_2\left(2^n\alpha-4\alpha-2^n\right)\right)}{2^n.3^n(g_2-h_2)^2(1+\alpha)^2(g_3-h_3)(1+2\alpha)},$$

Also, from **lemma 1.1.5**, notice, let $c_2 = a_3$ and $c_1 = a_2$.

$$a_2^2 = -\frac{b^2 p_1^2}{2^{2n}(g_2-h_2)^2(1+\alpha)^2},$$

$$a_3 - \mu a_2^2 = \frac{-p_2}{3^n(1+2\alpha)(g_3-h_3)} + \frac{7p_1^2 b}{3^n.4(1+2\alpha)(g_3-h_3)} +$$
$$\frac{b^2 p_1^2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^2))}{2^n.3^n(1+\alpha)^2(g_2-h_2)^2(1+2\alpha)(g_3-h_3)} - \mu\frac{b^2 p_1^2}{2^{2n}(1+\alpha)^2(g_2-h_2)^2}$$

$$a_3 - \mu a_2^2 = \frac{-p_2[4.2^{2n}(1+\alpha)^2(g_2-h_2)^2]+[7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 + 4.b^2 h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)) - \mu b^2 4.3^n(1+2\alpha)(g_3-h_3)]p_1^2}{4.2^{2n}.3^n(1+\alpha)^2(1+2\alpha)(g_2-h_2)^2(g_3-h_3)}$$

Now, let

$k = 3^n(1+2\alpha)(g_3-h_3)$ and

$$v = \frac{-[7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 + 4b^2h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)) - \mu b^2.4.3^n(1+2\alpha)(g_3-h_3)]}{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2}$$

So, $a_3 - \mu a_2^2 = c_2 - vc_1^2 = \dfrac{-b}{k}[p_2 - vp_1^2]$

More so, if $v \le 0$, then

$$\frac{[-7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 + 4b^2h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n)) + \mu b^2.4.3^n(1+2\alpha)(g_3-h_3)]}{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2} \le 0,$$

$$\mu b^2.4.3^n(1+2\alpha)(g_3-h_3) \le 7.b.2^{3n}(1+\alpha)^2(g_2-h_2)^2 - 4b^2h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))$$

$$\mu \le \frac{7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 - 4b^2h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))}{b^2.4.3^n(1+2\alpha)(g_3-h_3)}.$$

Hence, $$\mathrm{K}_1 = \frac{7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2 - 4b^2h_2(2^n g_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))}{b^2.4.3^n(1+2\alpha)(g_3-h_3)}. \qquad (33)$$

Also, if $v \ge 1$, then

$$\frac{[-7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2+4b^2h_2(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))+\mu b^2.4.3^n(1+2\alpha)(g_3-h_3)]}{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2}\geq 1$$

$$\mu b^2.4.3^n(1+2\alpha)(g_3-h_3)\geq 4.2^{2n}(1+\alpha)^2(g_2-h_2)^2+7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2-$$
$$4b^2h_2(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))$$

$$\mu\geq\frac{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2+7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2-4b^2h_2.(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))}{b^2.4.3^n(1+2\alpha)(g_3-h_3)}.$$

Hence, $$\mathrm{K}_2=\frac{4.2^{2n}(1+\alpha)^2(g_2-h_2)^2+7.b.2^{2n}(1+\alpha)^2(g_2-h_2)^2-4b^2h_2(2^ng_2(3\alpha+1)+h_2(2^n\alpha-4\alpha-2^n))}{b^2.4.3^n(1+2\alpha)(g_3-h_3)}.$$

(34)

By applying **lemma 1.1.6** with simple substitution of the various variables involves completes the desire result of prove of the theorem 2.2.1

Corollary 2.2.2: When $n=0$

Given that $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-...$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,$ $k\geq 2,\ \alpha\geq 0$, $n\in N_0$, ℓ is a function of sigmoid and $\ell'(0)>0,$ then for any real number μ:

$$\left|a_3-\mu a_2^2\right|\leq 2k$$

$$\begin{cases} 1-\dfrac{7.b(1+\alpha)^2(g_2-h_2)^2+4b^2h_2\left(g_2(3\alpha+1)-h_2(3\alpha+1)\right)-\mu b^2 4(1+2\alpha)(g_3-h_3)}{4(1+\alpha)^2(g_2-h_2)^2} & ,\text{if } \mu\leq \mathrm{K}_1 \\ 1 & ,\text{if } \mathrm{K}_1\leq\mu\leq\mathrm{K}_2 \\ \dfrac{7.b(1+\alpha)^2(g_2-h_2)^2+4b^2h_2\left(2^ng_2(3\alpha+1)+h_2\left(2^n\alpha-4\alpha-2^n\right)\right)-\mu b^2 4(1+2\alpha)(g_3-h_3)}{4(1+\alpha)^2(g_2-h_2)^2}-1 & ,\text{if } \mu\geq\mathrm{K}_2 \end{cases}$$

where

$$K=\frac{b}{(1+2\alpha)(g_3-h_3)}$$

$$\mathrm{K}_1=\frac{7.b(1+\alpha)^2(g_2-h_2)^2-2^2b^2h_2\left(g_2(3\alpha+1)+h_2(3\alpha+1)\right)}{2^2b^2(1+2\alpha)(g_3-h_3)},$$

$$K_2=\frac{4(g_2-h_2)^2(1+\alpha)^2+7(g_2-h_2)^2(1+\alpha)^2-4h_2b^2(g_2(3\alpha+1)+h_2(3\alpha+1))}{4(g_3-h_3)(1+2\alpha)b^2}.$$

Corollary 2.2.3: when $\alpha=0$

Given that $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k > 0,\ h_k > 0,\ g_k - h_k > 0,\ k \geq 2,\ \alpha \geq 0,\ n \in N_0,\ \ell$ is a function of sigmoid and $\ell'(0) > 0,$ then for any real number μ:

$$\left|a_3 - \mu a_2^2\right| \leq 2k \begin{cases} 1 - \dfrac{7.b.2^{2n}(g_2 - h_2)^2 + 4b^2 h_2 2^n (g_2 - h_2) - \mu b^2 4.3^n (g_3 - h_3)}{4.2^{2n}(g_2 - h_2)^2}, & \text{if } \mu \leq \mathrm{K}_1 \\ 1, & \text{if } \mathrm{K}_1 \leq \mu \leq \mathrm{K}_2 \\ \dfrac{7.b.2^{2n}(g_2 - h_2)^2 + 4b^2 h_2 2^n (g_2 - h_2) - \mu b^2 4.3^n (g_3 - h_3)}{4.2^{2n}(g_2 - h_2)^2} - 1, & \text{if } \mu \geq \mathrm{K}_2 \end{cases}$$

where

$$K = \frac{b}{3^n (g_3 - h_3)}$$

$$\mathrm{K}_1 = \frac{7.b.2^{2n}(g_2 - h_2)^2 - b^2 h_2 2^{n+2}(g_2 - h_2)}{2^2.3^n b^2 (g_3 - h_3)},$$

$$K_2 = \frac{4.2^{2n}(g_2 - h_2)^2 + 2^{2n}.7(g_2 - h_2)^2 - h_2 b^2 2^{n+2}(g_2 - h_2)}{4.3^n (g_3 - h_3) b^2}.$$

Corollary 2.2.4: when $\alpha = 0, n = 0$

Given that $\ell(z) = 1 - 2z + 5z^2 - \frac{22}{3}z^3 + \frac{95}{12}z^4 - \ldots$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k > 0,\ h_k > 0,\ g_k - h_k > 0,\ k \geq 2,\ \alpha \geq 0,\ n \in N_0,\ \ell$ is a function of sigmoid and $\ell'(0) > 0,$ then for any real number μ:

$$\left|a_3 - \mu a_2^2\right| \leq 2k \begin{cases} 1 - \dfrac{7.b(g_2 - h_2)^2 + 4b^2 h_2 (g_2 - h_2) - \mu b^2 4(g_3 - h_3)}{4(g_2 - h_2)^2}, & \text{if } \mu \leq \mathrm{K}_1 \\ 1, & \text{if } \mathrm{K}_1 \leq \mu \leq \mathrm{K}_2 \\ \dfrac{7.b(g_2 - h_2)^2 + 4b^2 h_2 (g_2 - h_2) - \mu b^2 4(g_3 - h_3)}{4(g_2 - h_2)^2} - 1, & \text{if } \mu \geq \mathrm{K}_2 \end{cases}$$

where

$$K=\frac{b}{(g_3-h_3)}$$

$$\mathrm{K}_1=\frac{7.b(g_2-h_2)^2-4b^2h_2(g_2-h_2)}{4b^2(g_3-h_3)},$$

$$K_2=\frac{4(g_2-h_2)^2+7(g_2-h_2)^2-4h_2b^2(g_2-h_2)}{4(g_3-h_3)b^2}.$$

2.3 Fekete-Szego functional for the class $W_{n,h}^{\alpha,g}(b,\ell)$

Theorem 2.3.1

Let $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-\ldots$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,\ k\geq 2,\ \alpha\geq 0,\ n\in N_0,\ \ell$ is a function of sigmoid and $\ell'(0)>0,$ then for any real number μ:

$$\left|a_3-\mu a_2^2\right|\leq\frac{2^{1-n}}{3^n(1+2\alpha)(g_3-h_3)}\max\left\{1,\left|\frac{\begin{array}{l}7.b2^{2n+1}(1+\alpha)^2(g_2-h_2)^2+\\4b^2h_2\left(2^n g_2(3\alpha+1)+h_2\left(2^n\alpha-4\alpha-2^n\right)\right)\\-\mu b^2 4.3^n(1+2\alpha)(g_3-h_3)-2^{2n+2}(1+\alpha)^2(g_2-h_2)^2\end{array}}{2^{2n+2}(1+\alpha)^2(g_2-h_2)^2}\right|\right. \qquad (35)$$

This result is sharp.

Corollary 2.3.2: when $n=0$

Given that $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-\ldots$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,\ k\geq 2,\ \alpha\geq 0,\ n\in N_0,\ \ell$ is a function of sigmoid and $\ell'(0)>0,$ then for any real number μ:

$$\left|a_3-\mu a_2^2\right|\leq\frac{2}{(1+2\alpha)(g_3-h_3)}\max\left\{1,\left|\frac{\begin{array}{l}7.b2(1+\alpha)^2(g_2-h_2)^2+\\4b^2h_2\left(g_2(3\alpha+1)-h_2(3\alpha+1)\right)\\-\mu b^2 4(1+2\alpha)(g_3-h_3)-4(1+\alpha)^2(g_2-h_2)^2\end{array}}{4(1+\alpha)^2(g_2-h_2)^2}\right|\right.$$

This result is sharp.

Corollary 2.3.3: when $\alpha=0$

Given that $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-...$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,\ k\geq 2,\alpha\geq 0,\ n\in N_0,\ \ell$ is a function of sigmoid and $\ell'(0)>0,$ then for any real number μ:

$$\left|a_3-\mu a_2^2\right|\leq\frac{2^{1-n}}{3^n(g_3-h_3)}\max\left\{1,\left|\frac{\begin{array}{l}7.b2^{2n+1}(g_2-h_2)^2+\\ b^2h_2.2^{n+2}(g_2-h_2)\\ -\mu b^2 4.3^n(g_3-h_3)-2^{2n+2}(g_2-h_2)^2\end{array}}{2^{2n+2}(g_2-h_2)^2}\right|\right.$$

This result is sharp.

Corollary 2.3.4: when $\alpha=0,n=0$

Given that $\ell(z)=1-2z+5z^2-\frac{22}{3}z^3+\frac{95}{12}z^4-...$ and $f(z)$ is of the form eqn 1 that belongs to the class $W_{n,h}^{\alpha,g}(b,\ell)$ where b is a non-zero complex number, $g_k>0,\ h_k>0,\ g_k-h_k>0,\ k\geq 2,\alpha\geq 0,\ n\in N_0,\ \ell$ is a function of sigmoid and $\ell'(0)>0,$ then for any real number μ:

$$\left|a_3-\mu a_2^2\right|\leq\frac{2}{(g_3-h_3)}\max\left\{1,\left|\frac{\begin{array}{l}7.b.2(g_2-h_2)^2+\\ b^2h_2.4(g_2-h_2)\\ -\mu b^2 4(g_3-h_3)-4(g_2-h_2)^2\end{array}}{4(g_2-h_2)^2}\right|\right.$$

This result is sharp.

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