Computational Analysis of Fractional Integro-Differential Equations using Chebyshev series solution of second kind

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Abstract:
A new numerical scheme via a Chebyshev series method is used to solve a family of linear fractional integro-differential equations especially the Fredholm and Volterra equations. The linear fractional integro-differential equation can be transformed into a system of equations for the unknown function itself and its m derivatives by taking into account the mth order Chebyshev series of the unknown function at any point. Using MATLAB 2009 software, this approach provides a straightforward and closed-form solution to a linear fractional integro-differential equation. Finally, examples are provided to illustrate comparisons between the proposed Chebyshev series solution and some existing methods in this direction. The results obtained performed better in terms of quick convergence and stability over the existing Taylor series expansion methodologies.

Keywords: **Chebyshev series, Fredhvolm equations, fractional, Integro-differential, Volterra**

1. Introduction

DUE to their widespread use, numerical studies on fractional differential equations have been very influential [1]. Since we lack the separation of variables method and product rules that are often available for differential equations of integer order, it has proven difficult to compute the solution to fractional differential equations [2]. Different approaches to computing the solution have been put forth by numerous scholars in the field of fractional differential equations and fractional integro-differential equations. Fractional calculus and fractional differential equations have both attracted an increasing amount of attention lately. Investigations into the existence and distinctiveness/uniqueness of fractional differential equations solutions have been conducted [3, 4]. Transform method [5], homotopy analysis [6], Adomian decomposition method [7], variational iteration method [8], homotopy perturbation method [8], collocation method [9, 10]; and eulerian polynomial basis functions [11] are just a few to these. The iterated Galerkin methods are found in [12]. Additionally, mixed interpolation collocation techniques have been found in [13] for first order and second-order Volterra linear integro-differential equations. The interpolation collocation method was created by Hu [14] to solve Fredholm linear integro-differential equations. Rashed addressed a particular kind of integro-differential equation with integral derivatives [15]. Chebyshev wavelet of second order were used by Setia et. al [16] to solve the universal Fredholm-Volterra fractional Integro-Differential equation with nonlocal boundary conditions.

Additionally, Kanwal and Liu [17] introduced the Taylor expansion method for resolving Volterra integral equations, and Sezer [18, 19] expanded this approach to include both Volterra integral equations and differential equations. When solving linear Volterra-Fredholm integro-differential equations, Yalcinbas and Sezer's [20, 21] method was extended to include nonlinear Volterra-Fredholm integral equations. They also used this method to solve a high-order linear differential equation system

In this paper, we analyze a class of fractional derivatives linear integro-differential equations of the form

$$D^{\alpha} y(t) = p(t)y(t) + f(t) + \lambda_1 \int_0^1 K_1(t,x)y(x)dx + \lambda_2 \int_0^1 K_2(t,x)y(x)dx, \quad t \in I = [0,1]$$

(1)

subjected to the initial conditions

$y^{(k)}(0) = c_k$, $k = 0, 1, \ldots, n-1$, $n-1 < \alpha \le n$, $n \in N$, (2)

where $y^{(k)}(t)$ stands for the kth-order derivative of y(t). These linear integro-differential equations are generalizations of the Volterra, Fredholm, and linear fractional differential equations. These kinds of equations appear in numerous mathematical physics modeling issues, such as heat conduction in memory-containing materials. These kinds of equations appear in a variety of mathematical physics, modeling issues, including heat conduction in memory-rich materials. These equations are also used in issues involving conduction, convection, and radiation all together. The aforementioned equation reduces to a linear fractional differential equation when $\lambda_1 = 0$, $\lambda_2 = 0$. Additionally, Eq. (1) becomes a linear integro-differential equation for N, and numerous writers have examined this equation's numerical solutions in great detail.

In this study Eq. (1) shall be converted into a system of linear equations in the unknown function and its derivatives by using the Chebyshev series of the unknown function at any location with the aim to demonstrate the efficiency and effectiveness of the desired results. The mth-order approximation of the desired solution, which is exact for a polynomial of degree less than or equal to m, can be obtained by roughly solving the resulting system. Four examples were utilized to demonstrate the precision and efficacy of the suggested strategy in the end.

2. MOTIVATION

The family of linear fractional integro-differential equation that involve the Fredholm and Volterra equations have been extensively explored by using so many numerical schemes, such as the Taylor series expansion, Homotopy analysis method just to mention few out of many schemes. However, the Chebyshev series in this study aims to provide interesting solutions that help to address the quest for quick convergence and stability.

2.1 Basic definitions

Following are some fundamental definitions and characteristics of the second-kind Chebyshev polynomial and fractional calculus:

Definition 1: The second kind of Chebyshev polynomials $T_i(x)$ are defined by

$$T_0(x)=1,\ T_1(x)=2x-1,\ \ T_2(x)=8x^2-8x+1,$$
$$\vdots$$
$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x) \tag{3}$$

where n = 1, 2, . . .

These are orthogonal polynomials to the weight function $w(x)=\sqrt{1-x^2}$ on [0, 1]. The dilated and translated weight function can be defined as

$$w_n^k(x)=w(2^k x-2n+1) \tag{4}$$

for a given value of k and n appearing in the second kind of Chebyshev series expansion.

A function $w(x)\in L^2_{w_n^k}[0,\,1]$ is expanded as

$$w(x)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}c_{n,i}\,,\ \psi_{n,i}(x) \tag{5}$$

where $c_{n,i}=(u(x),\ \psi_{n,i}(x))$ in which $\langle .\,,.\rangle$ denotes the inner product $w(x)\in L^2_{w_n^k}[0,\,1]$.

The truncated series of [5] can be defined by

$$u(x)=\sum_{n=1}^{2^{k-1}}\sum_{i=0}^{R-1}c_{n,i}\,,\ \psi_{n,i}(x)=C^T\psi(x) \tag{6}$$

where

$C=\left[c_1,c_2,\ldots c_{2^{k-1}}\right]^T,\ \psi=\left[\psi_1,\psi_2,\ldots\psi_{2^{k-1}}\right]^T,\ c_i=\left[c_{i,0},c_{i,1},\ldots c_{i,R-1}\right]^T,\ \psi_i=\left[\psi_{i,0},\psi_{i,1},\ldots\psi_{i,R-1}\right]^T$

for $i=1,2,\ldots.2^{k-1}$

Definition 2 [21] D^q (q > 0 real) represents the fractional differential operator of order q in terms of Riemann–Liouville, defined by [3]

$$D^q y(t)=\begin{cases}\dfrac{1}{\Gamma(n-q)}\dfrac{d^n}{dt^n}\displaystyle\int_0^1\dfrac{y(x)}{(t-x)^{q-n+1}}dx, & 0\le n-1<q<n\\ \dfrac{d^n y(t)}{dt^n} & q=n\end{cases}$$

(7)

where n ∈ N, Γ (∗) denotes the Gamma function. When q = n, the fractional differential reduces to the ordinary nth derivative of y(t) to t.

Definition 3.[21] I^q denotes the fractional integral operator of order q in the sense of Riemann–Liouville, defined as

$$D^{-q}y(t)=I^q y(t)=\begin{cases}\dfrac{1}{\Gamma(q)}\displaystyle\int\dfrac{y(x)}{(t-x)^{1-q}}dx, & q>0\\ y(t), & q=0\end{cases} \tag{8}$$

Basic properties of the fractional operator are listed below [21]: for f ∈ Cα, α ≥ −1, μ ≥ 1, η ≥ 0, β > −1:

1. $I^\mu \in C_0$,
2. $I^\eta I^\delta f(x) = I^\delta I^\eta f(x)$,
3. $I^\delta I^\eta f(x) = I^{\delta+\eta} f(x)$,
4. $D^\delta D^\eta f(x) = D^{\delta+\eta} f(x)$,
5. $D^\delta I^\delta f(x) = f(x)$,
6.

$$I^\delta D^\delta f(x)=f(x)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0^+),\qquad m-1<\delta<m,\ m\in\mathrm{N}$$

Furthermore,

$$I^\delta x^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)}x^{k+\beta}$$

3. Fredholm integro-differential equations with fractional derivative

The class of Fredholm integro-differential equations with fractional derivatives of the type

$$D^\alpha y(t)=p(t)y(t)+f(t)+\lambda\int_0^1 K_1(t,x)y(x)dx,\ t\in I=[0,1]$$

(9)

subjected to the initial conditions

$y^{(k)}(0) = c_k$, k = 0, 1, . . . , n − 1, n − 1 < α ≤ n, n ∈ N, (10)

where c_k and λ are constants, f (t), p(t), and K(t, x) are given functions satisfying certain conditions which implies that Eq. (9) has a unique solution, and y(t) is an unknown function to be determined.

To obtain the solution to Eq. (9), we integrate both sides of Eq. (9) to t for n times. With the properties enlisted above, the equation is in the form

$$I^{n-\alpha}y(t)=I^n(p(t)y(t))+I^n f(t)+\lambda I^n\left(\int_0^1 K(t,x)y(x)dx\right)$$

(11)

Furthermore,

$$\int_0^1\frac{(t-x)^{n-\alpha-1}}{\Gamma(n-\alpha)}y(x)dx=\int_0^t\frac{(t-x)^{n-1}p(x)}{(n-1)!}y(x)dx+\int_0^t\frac{(t-x)^{n-1}f(x)}{(n-1)!}dx$$
$$+\frac{\lambda}{(n-1)!}\int_0^1 y(x)\int_0^t K(s,\,x)(x-s)^{n-1}dsdx+Q_n(t)$$

(12)

Next, we assume that the arrived solution y(x) is m+1 times continually differentiable on the interval I, i.e., $y \in C^{m+1}$. Consequently, for $y \in C^{m+1}$, y (x) can be represented in terms of the mth-order Chebyshev series as

$$y(x) = y(t)+y'(t)(x-t)+\cdots+y^{(m)}(t)\frac{(x-t)^m}{m!} + y^{(m+1)}(\xi)\frac{(x-t)^{m+1}}{(m+1)!}$$

(13)

where ξ is between x and t, which readily shown that the Lagrange remainder

$y^{(m+1)}(\xi)(x - t)m+1/(m + 1)!$ is sufficiently small for a large enough m provided that $y^{(m+1)}(x)$ is a uniformly bounded function for any m on the interval I. Due to this fact, we will neglect the remainder and the truncated Chebyshev series y(x) as

$$y(x)=\sum_{j=0}^{m} a_j T_j(x) \quad (14)$$

The remainder vanishes for a polynomial of degree less than or equal to m, means that mth-order Chebyshev series is exact. Substituting the approximate expression (14) for y(x) into Eq. (12), to have

$$\sum_{i=0}^{m}\int_0^t \frac{(t-x)^{n-\alpha-1}}{\Gamma(n-\alpha)} a_j T_j(x)dx = \sum_{i=0}^{m}\int_0^t \frac{(t-x)^{n-1}p(x)}{(n-1)} a_j T_j(x)dx + \int_0^t \frac{(t-x)^{n-1}f(x)}{(n-1)!}dx + \sum_{i=0}^{m}\frac{\lambda}{(n-1)!}\int_0^t a_j T_j(x) \int_0^t K(s,x)(x-s)^{n+1}dsdx + Q_n(t) \quad (15)$$

or further

$$k_{00}(t)y(t)+k_{01}(t)y'(t)+\cdots+k_{0m}(t)y^m(t) = f^{(n)}(t) \quad (16)$$

where

$$k_{0i}(t)=\frac{(-1)^i x^{(n+i+1)}}{(n+i-\alpha)\Gamma(n-\alpha)!} - \frac{\lambda}{(n-1)!i!}\int_0^t (x-t)^i \int_0^t K(s,x)(x-s)^{n-1}dsdx - \frac{(-1)^i}{(n-1)!i!}\int_0^t p(x)(t-x)^{n+i-1}dx, \quad i=0,1,2,\ldots,m \quad (17)$$

$$f_{(n)}(t) = \frac{1}{(n-1)!}\int_0^t (t-x)^{n-1} f(x)dx+Q_n(t) \quad (18)$$

Thus Eq. (12) becomes an mth-order, linear, ordinary differential equation with variable coefficients for y(t) and its derivations up to m. We will determine y(t), . . . , $y^{(m)}(t)$ by solving linear equations. To this end, other m-independent linear equations for y(t), . . . , $y^{(m)}(t)$ are needed. This can be achieved by integrating both sides of Eq. (12) to t from 0 to s and changing the order of the integrations to have

$$\int_0^t \frac{(t-x)^{n-\alpha}}{\Gamma(n+1-\alpha)} y(x)dx = \int_0^t \frac{(t-x)^n p(x)}{n!} y(x)dx + \int_0^t \frac{(t-x)^n f(x)}{n!} + Q_n(x)dx + \frac{\lambda}{n!}\int_0^t y(x)\int_0^t K(t,x)(x-s)^n dsdx. \quad (19)$$

where we have replaced variable s with t, for convenience. Applying the Chebyshev series again and substituting (14) for y(x) into Eq. (19) gives

$k_{10}(t)y(t) + k_{11}(t)y'(t) + \cdot \cdot \cdot + k_{1m}(t)y(m)(t) = f_{(n+1)}(t)$, (20)

$$k_{ij}(t) = \frac{(-1)^j x^{n+j+1-\alpha}}{(n+j+1-\alpha)\Gamma(n+1-\alpha)j!} - \frac{\lambda}{n!j!}\int_0^1 (x-t)^j \int_0^t K(s,x)(x-s)^n dsdx - \frac{(-1)^j}{n!j!}\int_0^t p(x)(t-x)^{n+j}dx, \quad j=0,1,\ldots,m \quad (21)$$

$$f_{(n+1)}(t) = \int_0^t \frac{(t-x)^n f(x)}{n!} + Q_n(x)dx \quad (22)$$

Now arrived at another linear equation for $y^{(j)}(t)$, (j = 0, . . ., m) with $y^{(0)}(t) = y(t)$. By repeating the above integration process for i (i ∈ N+, 1 < i ≤ m) times, one can arrive at

$$k_{i0}(t)y(t) + k_{i1}(t)y'(t) + \ldots + k_{im}(t)y^m(t) = f_{(n+1)}(t); \qquad f \le m$$

(23)

where

$$k_{ij}(t) = \frac{(-1)^j y^{n+j+1-\alpha}}{(n+j+i-\alpha)\Gamma(n+i-\alpha)j!} - \frac{(-1)^j}{(n+i-1)!j!}\int_0^t p(x)(t-x)^{n+j+i-1}dx - \frac{\lambda}{(n+i-1-\alpha)!j!}\int_0^t (x-t)^j \int_0^t K(s,t)(x-s)^{n+i-1} dsdx, \quad (24)$$

$$f_{(r)}(t) = \int_0^t f_{(r-1)}(x)dx, \qquad t > n+1,\ r \in N^+ \quad (25)$$

Therefore, Eqs. (16), (20), and (23) form a system of m+1 linear equations for m+1 unknown functions y(t), . . ., $y^{(m)}(t)$.

For simplicity, this system can be rewritten in a Matrix form as

$K_{mm}(t)Y_m(t) = F_m(t)$, (26)

where $K_{mm}(t)$ is an (m + 1) × (m + 1) square matrix function in t, $Y_m(t)$ and $F_m(t)$ are two vectors of length m + 1, and these are defined as

$$K_{mm}(t) = \begin{pmatrix} k_{00}(t) & k_{01}(t) & \cdots & k_{0m}(t) \\ k_{10}(t) & k_{11}(t) & \cdots & k_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{m0}(t) & k_{m1}(t) & \cdots & k_{mm}(t) \end{pmatrix}$$

(27)

$$Y_m(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m)}(t) \end{pmatrix} \qquad F_m(t) = \begin{pmatrix} f_{(n)}(t) \\ f_{(n+1)}(t) \\ \vdots \\ f_{(n+m)}(t) \end{pmatrix} \quad (28)$$

If only y(t) is of concern, it can be solved easily with the aid of the well-known Cramer's rule. Hence, we can easily obtain the mth-order approximate solution as

$$y(t) = \frac{\Delta M_{mm}(t)}{\Delta K_{mm}(t)} \quad (29)$$

where

$$M_{mm}(t) = \begin{pmatrix} f_n(t) & k_{01}(t) & \cdots & k_{0m}(t) \\ f_{n+1}(t) & k_{11}(t) & \cdots & k_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+m}(t) & k_{m1}(t) & \cdots & k_{mm}(t) \end{pmatrix} \quad (30)$$

Remark 1. The above method is readily extended to solve the following integro-differential equation

$$\sum_{ii=0}^{N} P_i(t)D^{\alpha^i} y(t) = f(t) + \sum_{i=0}^{M} \lambda_i \int_0^1 K_i(x,t)y^{(i)}(x)dx.$$

(31)

4. Volterra integro-differential equations with fractional derivative

A class of Volterra integro-differential equations with fractional derivatives of the form

$$D^\alpha y(t) = p(t)y(t) + f(t) + \lambda\int_0^t K(t,x)y(x)dx \quad (32)$$

subjected to the initial conditions

$y^{(k)}(0) = c_k$, k = 0, 1, . . . , n − 1, n − 1 < α ≤ n, n ∈ N.

(33)

First, we integrate both sides of Eq. (32) to t for n times. With the aid of the property, we can get the equation in the following form

$$\int_0^t \frac{(t-x)^{n-\alpha-1}}{\Gamma(n-\alpha)} y(x)dx = \int_0^t \frac{(t-x)^{n-1} p(x)}{(n-1)!} y(x)dx + \int_0^t \frac{(t-x)^{n-1} f(x)}{(n-1)!}dx + \frac{\lambda}{(n-1)!}\int_0^t y(x)\int_x^t K(s,x)(x-s)^{n-1}dsdx + Q_n(t)$$

(34)

Substituting (14) for y(x) into Eq. (34), one can get

$a_{00}(t)y(t) + a_{01}(t)y'(t) + \cdot\ \cdot\ \cdot + a_{0m}(t)y^{(m)}(t) = f_{(n)}(t)$, (35)

where

$$a_{oj}(t) = \frac{(-1)^j x^{n+j-\alpha}}{(n+j-\alpha)\Gamma(n-\alpha)j!} - \frac{\lambda}{(n-1)!j!}\int_0^t (x-t)^j \int_x^t K(s,x)(x-s)^{n-1}dsdx - \frac{(-1)^j}{(n-1)!j!}\int_0^t p(x)(t-x)^{n+j-1}dx, \qquad j = 0,1,\ldots,m$$

(36)

Using a procedure analogous to the previous section, we can obtain other m linear equations for y(t), y′(t), y′′(t), . . . , $y^{(m)}(t)$ as follows:

$a_{i0}(t)y(t) + a_{i1}(t)y'(t) + \cdot \cdot + a_{im}(t)y^{(m)}(t) = f_{(n+i)}(t), \quad i \le m,$ (37)

where

$$a_{ij}(t) = \frac{(-1)^j x^{n+j+i-\alpha}}{(n+j+i-\alpha)\Gamma(n+i-\alpha)\, j!} - \frac{(-1)^j}{(n+i-1)!\, j!}\int_0^t p(x)(t-x)^{n+j+i-1}dx - \frac{\lambda}{(n+i-1)!\, j!}\int_0^t (x-t)^j \int_x^t K(s,t)(x-s)^{n+i-1}ds dx \quad (38)$$

Eqs. (35) and (37) constitute a system of m + 1 linear equations for m + 1 unknown functions of y(t), y′(t), y′′(t), . . . , $y^{(m)}(t)$, which can be rewritten in an alternative compact form as

$A_{mm}(t)Y_m(t)= F_m(t)$, (39)

where $A_{mm}(t)$ is an (m + 1) × (m + 1) square matrix function in t, $Y_m(t)$ and $F_m(t)$ are two vectors of length m + 1, and these are defined as

$$A_{mm}(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & \cdots & a_{0m}(t) \\ a_{10}(t) & a_{11}(t) & \cdots & a_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0}(t) & a_{m1}(t) & \cdots & a_{mm}(t) \end{pmatrix} \quad (40)$$

$$Y_m(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m)}(t) \end{pmatrix} \quad F_m(t) = \begin{pmatrix} f_{(n)}(t) \\ f_{(n+1)}(t) \\ \vdots \\ f_{(n+m)}(t) \end{pmatrix} \quad (41)$$

Using Cramer’s formula, we can easily obtain y(t). Hence we can easily obtain the mth-order approximate solution as

$$y(t) = \frac{\Delta M_{mm}(t)}{\Delta K_{mm}(t)} \quad (42)$$

where

$$M_{mm}(t) = \begin{pmatrix} f_n(t) & a_{01}(t) & \cdots & a_{0m}(t) \\ f_{n+1}(t) & a_{11}(t) & \cdots & a_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+m}(t) & a_{m1}(t) & \cdots & a_{mm}(t) \end{pmatrix} \quad (43)$$

Remark 2. The following integro-differential equation can be easily solved using the previously mentioned method.

$$\sum_{i=0}^{N} P_i(t)D^{\alpha^i} y(t) = f(t) + \sum_{i=0}^{M} \lambda_i \int_0^t K_i(x,t)y^{(i)}(x)dx. \quad (44)$$

5. Numerical examples

Several numerical examples are used in this part to demonstrate the value of the approach suggested in this study. The absolute errors as $|y_m(t)-y(t)|$ between the corresponding precise value y(t) and the m th order approximation values $y_m(t)$ are calculated in the computations that follows for convenience, it is also simple to obtain approximation of higher order. Additionally, the relevant outcomes are more precise.

Example 1. Consider a fractional differential equation [22].

$$D^{1.5}y(t) = 12t - 6t^{0.5}, \quad (45)$$

The exact answer and new result at evenly spaced mesh locations in [0, 1] are computed using the Chebyshev series of second kind, which also used third-order and sixth-order approximations. Table 1 displays the corresponding numerical results. The accuracy is admirably good. Moreover, using higher-order approximations will result in greater accuracy.

subjected to

$y(0) = 0, y'(0) = 0,$ (46)

and the exact solution is $y(t) = 12\Gamma(2)t^{2.5}/\Gamma(3.5) - 6\Gamma(1.5)t^2/\Gamma(3)$.

TABLE 1
ABSOLUTE ERRORS OF EXAMPLE 1

T	Existing m = 3[22]	Proposed m = 3	Existing m = 6[22]	Proposed m = 6
0.2	0.0002	0.0001	0.0001	0.0001
0.4	0.0012	0.0010	0.0000	0.0000
0.6	0.0033	0.0028	0.0000	0.0000
0.8	0.0067	0.0054	0.0001	0.0000
1.0	0.0117	0.0104	0.0002	0.0001

The exact answer and new result at evenly spaced mesh locations in [0, 1] are computed using the Chebyshev series of second kind, which also used third-order and sixth-order approximations. Table 1 displays the corresponding numerical results. The accuracy is admirably good. Moreover, using higher-order approximations will result in greater accuracy.

Example 2. Consider the Bagley–Torvik equation [23]

$$[D^2 + D^{3/2} + D^0]y(t) = t^2 + 2 + 4t^{0.5}/1.7725, \quad (47)$$

with the conditions

$$y(0) = 0,\ y'(0) = 0. \quad (48)$$

The exact solution is $y(t) = t^2$.

TABLE 2
ABSOLUTE ERRORS OF EXAMPLE 2

T	m = 3	Existing [22]	Proposed
0.1	0	0.0100	0.0100
0.2	0	0.0400	0.0400
0.3	0	0.0900	0.0900
0.4	0	0.1600	0.1600
0.5	0	0.2500	0.2500
0.6	0	0.3600	0.3600
0.7	0	0.4900	0.4900
0.8	0	0.6400	0.6400
0.9	0	0.8100	0.8100
1.0	0	1.0000	1.0000

In Table 2, there is a comparison of the approximate using Chebyshev series of second kind and exact solutions which is the existing method of Taylor series. The precision is excellent, and the third-order approximation simplifies the precise answer. As the mth-order approximation reduces to the exact solution if the precise answer is a polynomial of degree less than or equal to m, which is what we anticipated.

Example 3. Consider the Fredholm integro-differential equation with a parameter [22].

$$\lambda y' + \int_0^t \cos(\pi t x)\, y(x)\, dx = f(t) \quad (49)$$

$$y(0) = 1. \quad (50)$$

In the above equation, λ is taken as 1 or 0.01, and f (t) is chosen such that the exact solution is $y(t) = 1 + t^2$. The method to find the first- and second-order approximations is discussed in this study.

TABLE 3
ABSOLUTE ERRORS OF EXAMPLE 3

X	Existing λ=1, m= 1[22]	New Result λ=1, m= 1	Existing λ = 0.01, m = 1[22]	New Result λ = 0.01, m = 1	m = 2
0.1	0.0162	0.0149	0.0510	0.0460	0
0.2	0.0228	0.0204	0.1502	0.1420	0
0.3	0.0236	0.0212	0.0473	0.0421	0
0.4	0.0227	0.0218	0.0627	0.0589	0
0.5	0.0251	0.0239	0.0808	0.0769	0
0.6	0.0361	0.0345	0.0714	0.0678	0
0.7	0.0610	0.0590	0.0388	0.0304	0
0.8	0.1033	0.0969	0.0169	0.0124	0
0.9	0.1641	0.1422	0.0969	0.0876	0
1.0	0.2415	0.2146	0.2025	0.1985	0

Table 3 showed the absolute differences between the Taylor series solution and its approximate (Chebyshev series of second kind). The first approximates accuracy is not very excellent, especially at t = 1 but second-order approximation, on the other hand, reduces to the precise solution. This is simple to understand because the precise solution is a polynomial of degree 2. In the context of the current study, the exact solution is essentially provided by the second-order approximation. This is unaffected by the size of λ.

Example 4. Consider an integro-differential equation with fractional derivative [22]

$$D^{0.75} g(t) + \frac{1}{5} t^2 e^t g(t) - \int_0^t e^t s g(s) ds = \frac{6t^{2.25}}{\Gamma(3.25)}, \quad (51)$$

subjected to $g(0) = 0$, (52)

and the exact solution is $g(t) = t^3$.

We assess the numerical outcomes using the approximate method for m = 1, 2, or 3.

TABLE 4
ABSOLUTE ERRORS OF EXAMPLE 4.

t	m = 1[22]	m = 1	m = 2[22]	m = 2	m = 3
0.2	0.0007	0.0007	0.0001	0.0001	0
0.4	0.0013	0.0013	0.0006	0.0006	0
0.6	0.0172	0.0172	0.0022	0.0022	0
0.8	0.0378	0.0378	0.0059	0.0059	0
1.0	0.0629	0.0629	0.0135	0.0135	0

Table 4 depicted the related absolute errors in comparison to the Chebyshev series of second kind results. The accuracy is admirably good. Moreover, using higher-order approximations will result in greater accuracy.

CONCLUSION

Fredholm and Volterra integro-differential equations are addressed in this study along with a straightforward and efficient method for solving a large class of linear integro-differential equations with fractional derivatives. Under the right circumstances, the integro-differential equations can be transformed into a system of linear equations for the functions and their derivatives by using the Chebyshev series expansion of the functions at every point. Solving the resulting system of linear equations, which may be efficiently computed using software (MATLAB 2009), will yield the necessary approximate answer. The examples demonstrate the excellent accuracy and simplicity of this procedure while comparing the approximate with the existing (Taylor) method and Eulerian Polynomial basis functions. The derived mth-order approximation is additionally accurate for polynomials with degrees equal to or below m.

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