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Financial Data and a New Generalization of the Skew-T Distribution

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***Abstract*:** This work introduces a three-parameter hybrid model named the exponentiated half logistic skew-t distribution using the exponentiated half logistic generalised distributions. The hybrid model is appropriate for modelling skewed, heavy-and-long-tail datasets. The theoretical properties of the new model were investigated. Simulation studies performed to evaluate the finite sample performance of the parameter estimates using selected true parameter values showed that the estimates approached the true values as the sample size increased. The hybrid model efficacy, applicability and flexibility were demonstrated using the Nigeria inflation rate dataset, and the result indicated that the hybrid model outperformed several competing distributions.

Keywords: Heavy-tail distribution, Inflation rates, Order statistics, Skew-t distribution, Simulation.

1.0 Introduction

The skew-t distribution, a skewed extension of the symmetric Student-t distribution based on introducing a skew parameter, is often used in different fields such as reliability, economy, finance and volatility analysis. A lot of authors; [1], [2], [3] and [4, 5] have introduced various forms of the skew-t distribution as seen in the literature. Also, several possible skew-t distribution generalisations like exponentiated skew-t by Dikko and Agboola [6], odd exponentiated skew-t distribution by Adubisi *et al.* [7], Balakrishnan skew-t distribution by Shafiei and Doostparast [8], generalised hyperbolic skew-t distribution by Aas and Haff [9], Kumaraswamy skew-t distribution by Khamis *et al*. [10], Beta skew-t distribution by Shittu *et al.* [11], type I half-logistic skew-t distribution by Adubisi *et al.* [12], and Beta skew-t distribution by Basalamah *et al*. [13] have been introduced.

However, [14] established a tractable skewed extension of the symmetric student-t distribution known as the skew-t distribution by introducing a scaling factor on the two degrees of freedom of the student-t density function introduced by Jones [15]. The cumulative distribution function (cdf) of the skew-t distribution is given by

$$G_{ST}(x)=\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right) \qquad (1)$$

$$x\in(-\infty,\infty)$$

The corresponding probability distribution function (pdf) is given by

$$g_{ST}(x)=\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}} \qquad (2)$$

where λ controls the skewness.

The cdf of the exponentiated half-logistic generalised distribution introduced by Cordeiro et al. [16] is given by

$$F(x)=\int_0^{-\log\left[1-G(x,\varsigma)\right]}\frac{2\alpha\beta e^{-\alpha x}\left[1-e^{-\alpha x}\right]^{\beta-1}}{\left(1+e^{-\alpha x}\right)^{\beta+1}}dx$$

$$=\left\{\frac{1-\left[1-G(x;\varsigma)\right]^{\alpha}}{1+\left[1-G(x;\varsigma)\right]^{\alpha}}\right\}^{\beta} \qquad (3)$$

and the corresponding pdf is given by

$$f(x)=2\alpha\beta g(x;\varsigma)\left[1-G(x;\varsigma)\right]^{\alpha-1}\times\frac{\left\{1-\left[1-G(x;\varsigma)\right]^{\alpha}\right\}^{\beta-1}}{\left\{1+\left[1-G(x;\varsigma)\right]^{\alpha}\right\}^{\beta+1}} \qquad (4)$$

where $\alpha,\beta>0$ are two additional shape parameters. $G(x;\varsigma)$ and $g(x;\varsigma)$ are the baseline cdf and pdf depending on a parameter vector ς.

In this article, a new generalisation of the skew-t distribution based on the exponentiated half-logistic-G distribution is introduced. The new class of distribution called the exponentiated half-logistic skew-t distribution is capable of fitting skewed, long-tail and heavy-tail datasets and is more flexible than the skew-t distribution as it contains the skew-t distribution and other important distributions as special cases. In distributional theory, heavy-tailed distributions are continuous probability distributions whose tails are not bounded exponentially, and all long-tailed distributions are heavy-tailed [17]. This article focuses on extending the applicability of the skew-t distribution by adding a parameter (shape) to increase its efficacy in modelling financial datasets. The motivation in developing the new distribution is to create a more flexible distribution with right-skewed, left-skewed, and unimodal features. The new distribution will analyse the inflation rates in sub-Sahara Africa, specifically Nigeria inflation rate. The new distribution can serve as an alternative error innovation in modelling and forecasting financial return series using GARCH models in future research studies.

The remaining part of this article is structured as follows: In Section 2, we define the exponentiated half-logistic skew-t (EHL_{ST}) distribution. Section 3 derives the series expansions of the EHL_{ST} density and distribution functions. Related statistical properties of the new distribution are presented in section 4. In Section 5, we derive the estimates of the unknown parameters using the maximum likelihood estimation procedure. The simulation study is conducted to assess the MLEs consistency and illustration of the EHL_{ST} flexibility and efficacy using the Nigeria inflation rates dataset. Conclusion in section 6.

2.0 The Exponentiated Half-Logistic Skew-T (EHL_{ST}) Distribution

The cdf of the three-parameter EHL_{ST} distribution derived by substituting (1) in (3) is given as

$$F(x,\alpha,\beta,\lambda)=\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}\right\}^{\beta} \tag{5}$$

and the corresponding pdf is given as

$$f(x,\alpha,\beta,\lambda)=2\alpha\beta\left(\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\right)\times\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha-1}\times\frac{\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta-1}}{\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta+1}}$$

$$\alpha>0,\quad \beta>0,\quad \lambda>0,\quad x\in(-\infty,\infty) \tag{6}$$

From now onward, we will denote a random variable X having pdf (6) by $X \sim EHL_{ST}(\xi)$, where $\xi=(\alpha,\beta,\lambda)$ are the set of parameters.

The survival function, hazard rate function (hrf), reversed hazard rate function (rhrf), cumulative hazard rate function (chrf) and odds function (Of) are given, respectively.

$$s(x,\xi)=1-\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}\right\}^{\beta} \tag{7}$$

The graphical structures of the EHL_{ST} distribution are illustrated using plots when the parameters are varied. To understand the effect of each parameter in determining the overall shape of the EHL_{ST} density function, some plots which depict the shape of the density curve are presented. Figure 1 depicts the shapes of the EHL_{ST} density function when the parameter α is varied, and the parameters β and λ are fixed. Figure 2 depicts the shapes of the EHL_{ST} density function when the parameter β is varied and the parameters α and λ are fixed. Figure 3, depicts the shape of the EHL_{ST} density function when the parameter λ is varied and the parameters α and β are fixed. More so, Figure 4 depicts the hazard function of the EHL_{ST} distribution at selected parameter values.

The plots indicate that the left and right tails get lighter and tend to zero as α and β approaches infinity, respectively while the EHL_{ST} density curve tend towards a flat curve as λ increases. The plots also reveal that the hazard rate function can be increasing, decreasing, and inverted bathtub shaped.

3.0 Series Expansions

In this section, we derive the series representations of the EHL_{ST} density and distribution functions. In order to

$$h(x,\xi)=\frac{2\alpha\beta\left(\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\right)\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha-1}\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta-1}}{\left(\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}-\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}\right)} \quad (8)$$

$$\tau(x,\xi)=\frac{2\alpha\beta\left(\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\right)\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha-1}\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta-1}}{\left(\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}\right)} \quad (9)$$

$$\mathrm{H}(x,\xi)=-\ln\left[1-\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}}\right\}^{\beta}\right] \quad (10)$$

$$O\left(x,\xi\right)=\frac{\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}}{\left(\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}-\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta}\right)} \quad (11)$$

obtain a simple form for the EHL_{ST} density, we expand (6) using the generalised binomial series representations:

$$\left(1+z\right)^{-\beta}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta+i-1}{i}z^i \quad |z|<1, \quad \beta>0 \quad (12)$$

$$\left(1-z\right)^{\beta-1}=\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{i}z^j \quad |z|<1 \quad \beta>0 \quad (13)$$

Let

$$A=\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{-(\beta+1)}$$

$$B=\left\{1-\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha}\right\}^{\beta-1}$$

By applying (12) in quantity A, gives

$$A=\sum_{i=0}^{\infty}(-1)^i\binom{\beta+i}{i}\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha i} \quad (14)$$

and (13) in quantity B, we have

$$B=\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\left[1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right)\right]^{\alpha j} \quad (15)$$

The EHL_{ST} density after some algebra can be written as

$$f(x;\xi)=\frac{\alpha\beta\lambda}{\left(\lambda+x^2\right)^{3/2}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\beta+i}{i}\binom{\beta-1}{j}\times\underbrace{\left[\frac{1}{2}\left(1-\frac{x}{\sqrt{\lambda+x^2}}\right)\right]^{\alpha(i+j+1)-1}}_{C} \quad (16)$$

By applying (13) in the quantity C, the expanded form of the EHL_{ST} density is given as

$$f(x;\xi)=w_{i,j,k}x^k\left(\lambda+x^2\right)^{-\left(\frac{k+3}{2}\right)} \quad (17)$$

where

$$w_{i,j,k}=\alpha\beta\lambda\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}}{2^{\alpha(i+j+1)-1}}\binom{\beta+i}{i}\binom{\beta-1}{j}\times\binom{\alpha(i+j+1)-1}{k}$$

Likewise, the expanded form of the EHL_{ST} distribution is given as

$$F(x;\xi)=\vartheta_{k,l}x^l\left(\lambda+x^2\right)^{-\frac{l}{2}} \quad (18)$$

where

$$\vartheta_{k,l}=-1+2^{\beta}\sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}}{2^{\alpha k}}\binom{\beta+k-1}{k}\binom{\alpha k}{l}$$

4.0 STATISTICAL PROPERTIES

In this section, some basic statistical properties of the EHL_{ST} distribution are examined using the established series expansions (mixture representations) of the density and distribution functions in Section 3.

4.1Quantile function

The quantile function, $Q(u)=F^{-1}(u)$ of X can be derived by inverting (5). The quantile function $Q(u)$ for $u\in(0,1)$ is given as

$$X=Q(u,\xi)=\lambda^{\frac{1}{2}}\frac{\left[1-2\left(\frac{1-u^{\frac{1}{\beta}}}{1+u^{\frac{1}{\beta}}}\right)^{\frac{1}{\alpha}}\right]}{\left[1-\left(1-2\left(\frac{1-u^{\frac{1}{\beta}}}{1+u^{\frac{1}{\beta}}}\right)^{\frac{1}{\alpha}}\right)^{2}\right]^{\frac{1}{2}}} \quad (19)$$

The median of X is obtained by setting $u=0.5$ in (19). Given that the uniform random variables are easily generated in most statistical software, the quantile function is useful in generating the EHLST random variables. The Bowley skewness [18] and Moors kurtosis [19] for EHL_{ST} can be examined by using the quantile function (17) as follows:

$$S_k=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \quad \text{and}$$

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \quad (20)$$

where $Q(.)$ is the EHL_{ST} quantile function. The Bowley skewness and Moors kurtosis measures do not depend on the moments of the distribution and are almost insensitive to outliers. Table 1, provides some numerical values of the median, 25th and 75th percentiles, interquartile range (IQR), skewness and kurtosis for some selected parameter values. It is observed that as λ increases across increasing values of α and β; the skewness, kurtosis, median, 25th and 75th percentiles while the IQR decreases.

4.2 Moment

In this subsection, we derive the r^th ordinary moment for the EHL_{ST} distribution. Let $X \sim EHL_{ST}(\xi)$ be a random variable then the r^th moment of X about the origin is expressed as

$$\mu_r = \mathrm{E}\left(X^r\right) = \int_{-\infty}^{+\infty} x^r f\left(x,\xi\right)dx \qquad (21)$$

By inserting (17) in (21), we have

$$\mathrm{E}\left(X^r\right) = \int_{-\infty}^{+\infty} x^r w_{i,j,k} x^k \left(\lambda + x^2\right)^{-\left(\frac{k+3}{2}\right)} dx \qquad (22)$$

The moment obtained after some algebra is given as

$$\mu_r' = \mathrm{E}\left(X^r\right) = w_{i,j,k}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+k+1}{2},\frac{2-r}{2}\right) \qquad (23)$$

and the characteristic function expressed as

$$\Phi_X\left(t\right) = w_{i,j,k}\sum_{r=0}^{\infty}\frac{\left(it\right)^r}{r!}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+k+1}{2},\frac{2-r}{2}\right) \qquad (24)$$

4.3 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are order statistics obtained from the sample. According to [20], the pdf, $f_{p:n}\left(x\right)$ of the p^th order statistics $X_{p:n}$ is defined as

$$f_{p:n}\left(x\right) = \frac{g\left(x\right)\left[G\left(x\right)\right]^{p-1}\left[1-G\left(x\right)\right]^{n-p}}{\mathrm{B}\left(p,n-p+1\right)} \qquad (25)$$

where, $G\left(x\right)$ and $g\left(x\right)$ are the cdf and pdf of the EHL_{ST} distribution respectively, and $\mathrm{B}(.,.)$ is the beta function. Since $0 < G\left(x\right) < 1$ for $x > 0$ using the binomial theorem (13) on $\left[1-G\left(x\right)\right]^{n-p}$, we have

$$f_{p:n}\left(x\right) = \frac{\sum_{l=0}^{n-p}\left(-1\right)^l\binom{n-p}{l}\left[G\left(x\right)\right]^{p+l-1}g\left(x\right)}{\mathrm{B}\left(p,n-p+1\right)} \qquad (26)$$

Therefore, inserting (5) and (6) in (26) and expanding based on (13), the p^th order statistics for EHL_{ST} distribution is given as

$$f_{p:n}\left(x\right) = \frac{\vartheta_{i,j,l,k,m}x^m\left(\lambda + x^2\right)^{-\left(\frac{3+m}{2}\right)}}{\mathrm{B}\left(p,n-p+1\right)} \qquad (27)$$

where,

$$\vartheta_{i,j,l,k,m} = \alpha\beta\lambda\sum_{l=0}^{n-p}\sum_{i=o}^{\beta(p+l)-1}\sum_{j,k,m=0}^{\infty}\frac{\left(-1\right)^{l+i+j+k}}{2^k}\binom{n-p}{l}\times\binom{\beta\left(p+l\right)-1}{i}\binom{\beta\left(p+l\right)+j}{j}\times\binom{\alpha\left(i+j+1\right)-1}{k}\binom{k}{m}$$

The distribution of the minimum and maximum order statistics can be obtained from (27) by setting $p=1$ and $p=n$.

Furthermore, the r^th moment of the p^th order statistics for EHL_{ST} distribution is defined as

$$\mathrm{E}\left(X_{p:n}^r\right) = \int_{-\infty}^{+\infty} x^r f_{p:n}\left(x;\xi\right)dx \qquad (28)$$

By inserting (27) in (28), we have

$$\mathrm{E}\left(X_{p:n}^r\right) = \frac{1}{\mathrm{B}\left(p,n-p+1\right)}\vartheta_{i,j,l,k,m}\int_{-\infty}^{+\infty} x^{r+m}\left(\lambda + x^2\right)^{-\left(\frac{3+m}{2}\right)}dx \qquad (29)$$

So, after some algebraic simplifications, the r^th moment of the p^th order statistics is given as

$$\mathrm{E}\left(X_{p:n}^r\right) = \frac{1}{\mathrm{B}\left(p,n-p+1\right)}\vartheta_{i,j,l,k,m}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+m+1}{2},\frac{2-r}{2}\right) \qquad (30)$$

4.4 Entropies

The entropy of a random variable X is a measure of the variation of uncertainty. According to [21], the Rényi entropy of a random variable with pdf $f(x)$ is given as

$$I_{R(\delta)} = \frac{1}{1-\delta}\log\int_{-\infty}^{+\infty} f(x)^{\delta}\,dx, \quad \delta > 0 \text{ and } \delta \neq 1 \tag{31}$$

By inserting (6) in (31), applying the binomial theorem and some algebraic simplification, the Rényi Entropy of EHL_{ST} distribution is given as

$$I_{R(\delta)} = \frac{1}{1-\delta}\log\left(\vartheta_{i,j,k,q}\lambda^{\frac{1-3\delta}{2}}\mathrm{B}\left(\frac{q+1}{2},\frac{3\delta-1}{2}\right)\right) \tag{32}$$

where,

$$\vartheta_{i,j,k,q} = (\alpha\beta\lambda)^{\delta}\sum_{i,j,k,q=0}^{\infty}\frac{(-1)^{i+j+k}}{2^k}\binom{\delta(\beta+1)+i-1}{i} \times\binom{\delta(\beta-1)}{j}\binom{\alpha(i+j+\delta)-\delta}{k}\binom{k}{q}$$

Furthermore, the q-entropy [22] is defined as

$$H_{\delta}(X) = \frac{1}{\delta-1}\log\left(1-\left[\int_{-\infty}^{+\infty} f(x)^{\delta}\,dx\right]\right), \quad \delta > 0 \text{ and } \delta \neq 0 \tag{33}$$

Therefore, the q-entropy of EHL_{ST} distribution is given as

$$H_{\delta}(X) = \frac{\log\left(1-\left(\vartheta_{i,j,k,q}\lambda^{\frac{1-3\delta}{2}}\mathrm{B}\left(\frac{q+1}{2},\frac{3\delta-1}{2}\right)\right)\right)}{\delta-1} \tag{34}$$

5.0 Parameter Estimation

5.1 Maximum Likelihood Estimation

The maximum likelihood estimation (MLEs) of the unknown parameters for EHL_{ST} distribution are determined based on the complete samples. Let $X_1, X_2, \ldots, X_n$ be the observed values from the EHL_{ST} distribution with unknown parameter vector $\xi = (\alpha,\beta,\lambda)^T$. The log-likelihood function, say l, of EHL_{ST} distribution is given as

$$l = n\ln\alpha + n\ln\beta + n\ln\lambda - 3/2\sum_{i=1}^{n}\ln\left(\lambda + x_i^2\right) + (\alpha-1)\sum_{i=1}^{n}\ln(z_i) + (\beta-1)\sum_{i=1}^{\infty}\ln\left(1-(z_i)^{\alpha}\right) - (\beta+1)\sum_{i=1}^{n}\ln\left(1+(z_i)^{\alpha}\right) \tag{35}$$

where $z_i = \left(1-\frac{1}{2}\left(1+\frac{x_i}{\sqrt{\lambda+x_i^2}}\right)\right)$

Taking the partial derivative of the log-likelihood l, with respect to α, β and λ equating to zero, the following normal equations are obtained as follows:

$$\frac{\partial(l)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(z_i) - (\beta-1)\sum_{i=1}^{n}\frac{(z_i)^{\alpha}\ln(z_i)}{\left\{1-(z_i)^{\alpha}\right\}} - (\beta+1)\sum_{i=1}^{n}\frac{(z_i)^{\alpha}\ln(z_i)}{\left\{1+(z_i)^{\alpha}\right\}} = 0 \tag{36}$$

$$\frac{\partial(l)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left(1-(z_i)^{\alpha}\right) - \sum_{i=1}^{n}\ln\left\{1+(z_i)^{\alpha}\right\} = 0 \tag{37}$$

and

$$\frac{\partial(l)}{\partial\lambda} = \frac{n}{\lambda} - \frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\lambda+x_i^2\right)} + (\alpha-1)\sum_{i=1}^{n}\frac{x}{4\left(\lambda+x_i^2\right)^{3/2}(z_i)} - \alpha(\beta-1)\sum_{i=1}^{n}\frac{x(z_i)^{\alpha}}{4\left(\lambda+x_i^2\right)^{3/2}(z_i)\left\{1+(z_i)^{\alpha}\right\}} - \alpha(\beta+1)\sum_{i=1}^{n}\frac{x(z_i)^{\alpha}}{4\left(\lambda+x_i^2\right)^{3/2}(z_i)\left\{1+(z_i)^{\alpha}\right\}} = 0 \tag{38}$$

The non-linear equations (36), (37) and (38) are solved numerically via iterative methods using statistical software such as R, MATLAB, Maple.

5.2 Simulation Study

In this section, we evaluate the performance of the maximum likelihood estimation (MLE) procedure for the EHL_{ST} model using the Monte Carlos simulation study. The performance of the estimators is evaluated through the average estimates (MEs), absolute bias, variance, mean square errors (MSE), and root mean square errors (RMSE) for different sample sizes. We generated 10,000 samples from the EHL_{ST} distribution, each sample size $n = 30, 50, 150, 250, 300, 1000$ for selected values $\xi = (\alpha, \beta, \lambda) = (0.5, 0.9, 0.5)$, $(1.2, 1.0, 0.7)$, $(1.5, 1.2, 1.0)$, and $(2.5, 1.0, 1.7)$. The absolute bias, MSE and RMSE are computed for $\hat{S} = \hat{\alpha}, \hat{\beta}, \hat{\lambda}$ using

$$
\begin{aligned}
Abs\hat{B}ias_s &= \left| \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{S}_i - S \right) \right| \\
\hat{M}SE_s &= \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{S}_i - S \right)^2 \qquad (39) \\
\hat{R}MSE_s &= \sqrt{\frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{S}_i - S \right)^2}
\end{aligned}
$$

The numerical results in Tables 2-5 shows that the MLEs are closer to the true values of the EHL_{ST} parameters. More so, the absolute bias, MSEs and RMSEs for each parameter decreases as the sample size increases. The simulations show the compatibility of the MLEs with the consistency property.

5.3 Application to the Inflation Rate Dataset

In this section, we provide an application using a real dataset to illustrate the flexibility, applicability and superiority of the EHL_{ST} distribution. The dataset is the Nigeria inflation rates as recorded by the central bank of Nigeria. This dataset contains 207 all items (year on change) inflation rates for each month from January 2003 to April 2020. The Nigeria inflation rates dataset can be sourced from the central bank of Nigeria web database at www.cbn.gov.ng.

Table 6: Descriptive statistics of the Nigeria inflation rates dataset.

N	Mean	SD	sk	Ks
207	11.96	4.27	0.92	1.55

Table 6 shows the descriptive statistics for the Nigeria inflation rate dataset. The number of observations indicators for the first four moments (mean, standard deviation (SD), skewness(sk), kurtosis(ks)) of the inflation rate dataset. The dataset is unimodal, moderately right-skewed, with a leptokurtic form of the histogram. Hence, the EHL_{ST} distribution can handle such a dataset given its shape properties, as depicted in Figures 1-4.

The purpose of the application is to compare the fitting performance of the EHL_{ST} distribution with some competitive distributions, i.e., based on extensions of the Fréchet, Pareto, Lomax, Burr xii and Weibull distributions. The following competitive distributions considered are the type-I half logistic skew-t ($TIHL_{ST}$) distribution [12], half logistic skew-t ($TIHL_{ST}$) distribution [23], skew-t (ST_2) distribution [14], exponentiated half-logistic Lomax (EHLL) distribution [24], exponentiated generalized skew-t (EGST) distribution [25], exponentiated Fréchet (EXF) distribution [26],

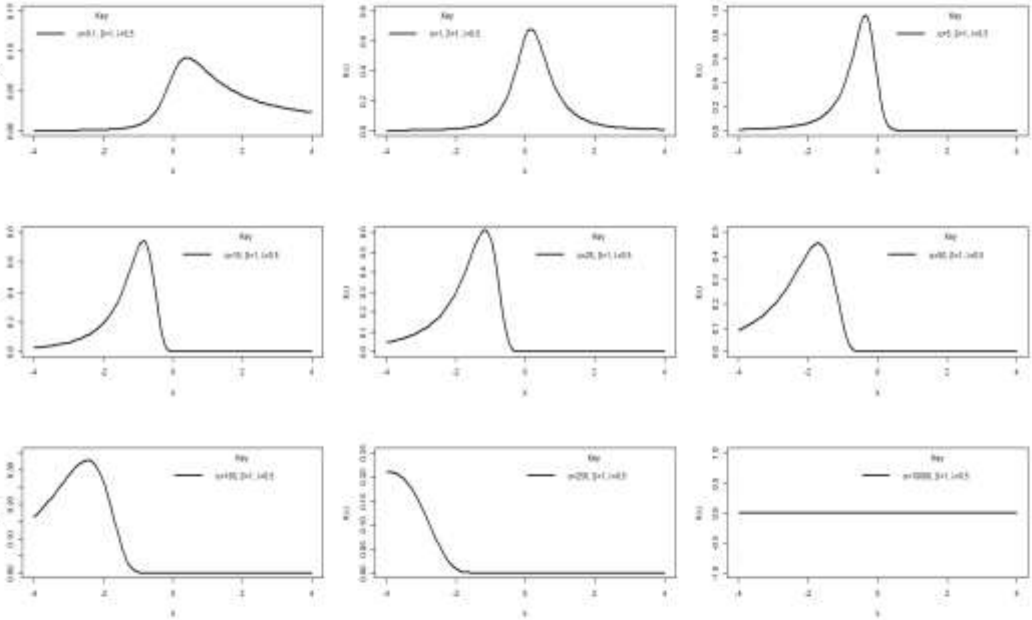

Figure 1: The EHL_{ST} density function (pdf) plots for some selected $(\alpha = \text{varied}, \beta = 1, \lambda = 0.5)$ parameter values.

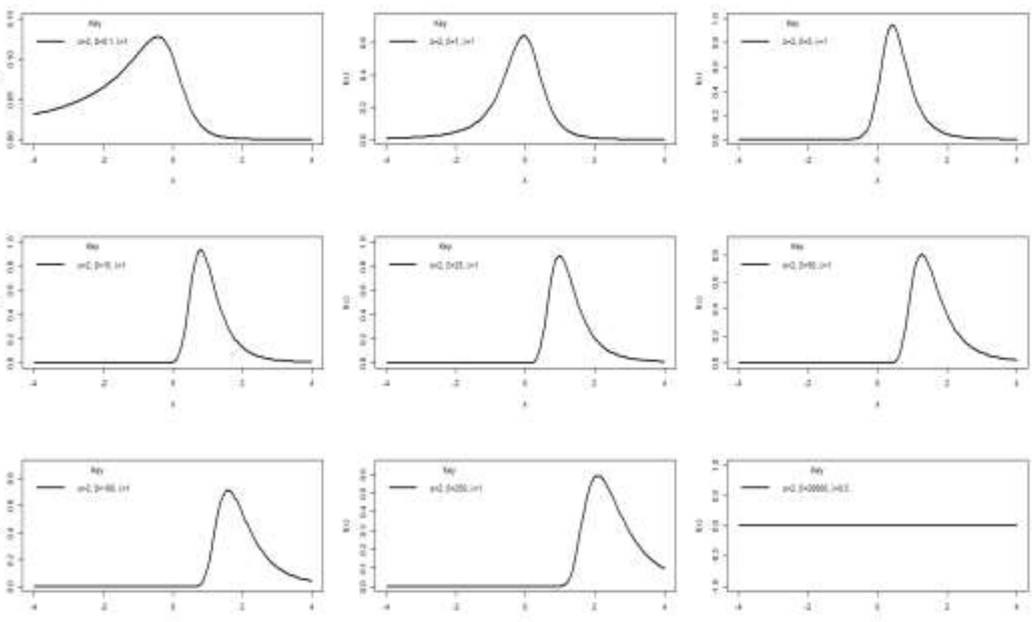

Figure 2: The EHL_{ST} density function (pdf) plots for some selected $(\alpha = 2, \beta = \text{varied}, \lambda = 1)$ parameter values.

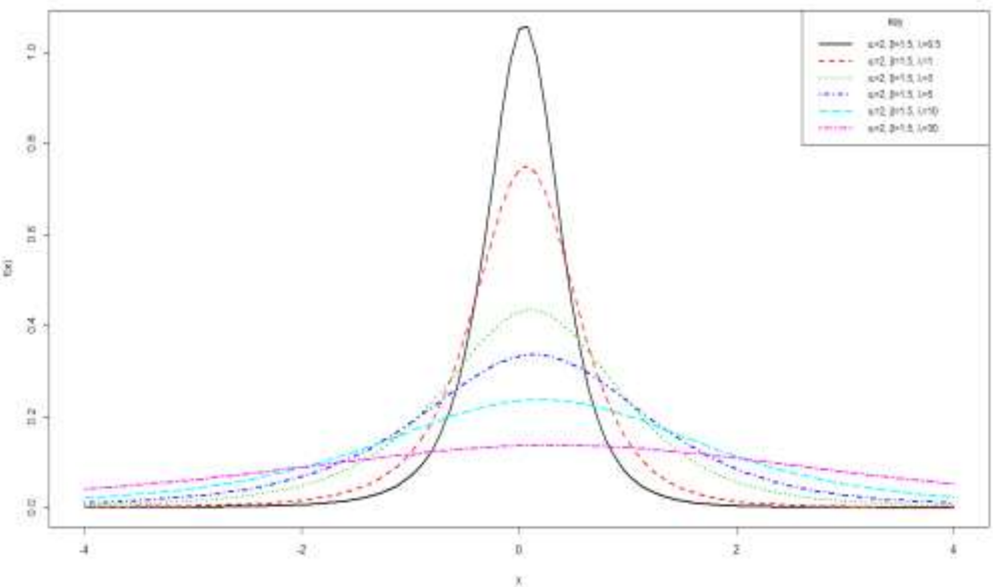

Figure 3: The EHL_{ST} density function (pdf) plots for some selected $(\alpha = 2, \beta = 1.5, \lambda = \text{varied})$ parameter values.

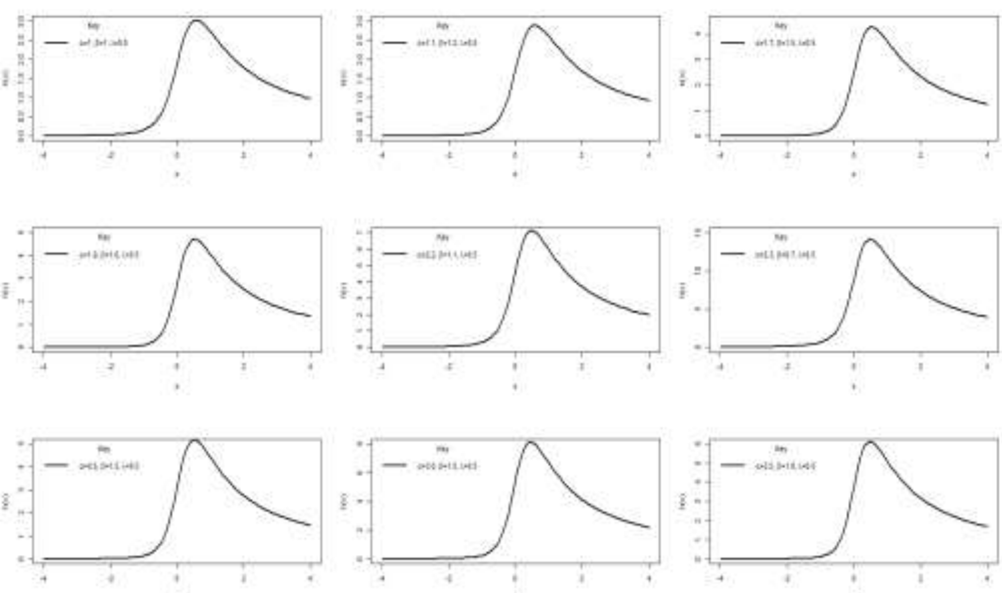

Figure 4: The EHL_{ST} hazard function plots for some selected $(\alpha = \text{varied}, \beta = \text{varied}, \lambda = 0.5)$ parameter values.

Table 1: Median (M), 25th and 75th percentiles, skewness (Sk), kurtosis (Ks) and IQR.

λ	α	β	**M**	**25th**	**75th**	**Sk**	**Ks**	**IQR**
0.5	0.3	0.2	2.1203	0.5822	9.0356	0.6361	3.0372	8.4534
	0.6	0.4	0.6549	0.1046	1.6828	0.3027	0.8756	1.5782
	0.7	0.6	0.5083	0.0254	1.2837	0.2325	0.6208	1.2583
	1.2	1.0	0.1439	-0.2278	0.5370	0.0281	0.0767	0.7648
	1.7	1.2	-0.0340	-0.3878	0.2757	-0.0665	-0.2205	0.6636
0.6	0.3	0.2	2.3227	0.6378	9.8980	0.6361	3.0372	9.2602
	0.6	0.4	0.7173	0.1146	1.8435	0.3027	0.8756	1.7289
	0.7	0.6	0.5568	0.0278	1.4062	0.2325	0.6208	1.3783
	1.2	1.0	0.1576	-0.2495	0.5883	0.0281	0.0767	0.8378
	1.7	1.2	-0.0372	-0.0425	0.3021	-0.0665	-0.2205	0.7269
0.7	0.3	0.2	2.5088	0.6889	10.6911	0.6361	3.0372	10.0022
	0.6	0.4	0.7749	0.1238	1.9912	0.3027	0.8756	1.8674
	0.7	0.6	0.6014	0.0301	1.5189	0.2325	0.6208	1.4888
	1.2	1.0	0.1702	-0.2695	0.6354	0.0281	0.0767	0.9050
	1.7	1.2	-0.0402	-0.4589	0.3263	-0.0665	-0.2205	0.7852
1.5	0.3	0.2	3.6725	1.0084	15.6501	0.6361	3.2119	14.6417
	0.6	0.4	1.1343	0.1812	2.9148	0.3027	1.0642	2.7336
	0.7	0.6	0.8804	0.0440	2.2234	0.2325	0.7841	2.1793
	1.2	1.0	0.2492	-0.3945	0.9302	0.0281	0.0767	1.3248
	1.7	1.2	-0.0589	-0.6718	0.4776	-0.0665	-0.2205	1.1494

Table 2: Simulation results for the EHL_{ST} distribution

$(\alpha = 0.5, \beta = 0.9, \lambda = 0.5)$						
n	**Par**	**ME**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	α	0.5690	0.0690	0.0445	0.0492	0.2219
	β	1.1566	0.2566	0.1489	0.2147	0.4634
	λ	0.7127	0.2127	0.5963	0.6415	0.8010
50	α	0.5378	0.0378	0.0192	0.0206	0.1436
	β	1.0851	0.1851	0.0628	0.0971	0.3116
	λ	0.6087	0.1087	0.2180	0.2298	0.4794
150	α	0.5108	0.0108	0.0043	0.0044	0.0662
	β	1.0249	0.1249	0.0140	0.0296	0.1720
	λ	0.5303	0.0303	0.0327	0.0336	0.1832
250	α	0.5066	0.0066	0.0024	0.0025	0.0497
	β	1.0144	0.1144	0.0081	0.0211	0.1454
	λ	0.5184	0.0184	0.0171	0.0174	0.1319
300	α	0.5054	0.0054	0.0020	0.0020	0.0448
	β	1.0119	0.1119	0.0066	0.0191	0.1384
	λ	0.5149	0.0149	0.0140	0.0143	0.1194
1000	α	0.5014	0.0014	0.0006	0.0006	0.0240
	β	1.0030	0.1030	0.0019	0.0125	0.1117
	λ	0.5046	0.0046	0.0039	0.0039	0.0623

Table 3: Simulation results for the EHL_{ST} distribution

$\left(\alpha = 1.2, \beta = 1.0, \lambda = 0.7\right)$

n	Par	ME	AbsBias	Var	MSE	RMSE
30	α	1.5449	0.3449	0.4415	0.5604	0.7486
	β	1.4410	0.4410	0.7140	0.9085	0.9532
	λ	1.2609	0.5609	1.3732	1.6878	1.2992
50	α	1.4442	0.2442	0.3010	0.3606	0.6005
	β	1.2964	0.2964	0.4115	0.4994	0.7067
	λ	1.0974	0.3974	0.8529	1.0108	1.0054
150	α	1.2904	0.0904	0.0899	0.0981	0.3132
	β	1.0979	0.0979	0.0867	0.0963	0.3103
	λ	0.8388	0.1388	0.1900	0.2093	0.4575
250	α	1.2542	0.0542	0.0478	0.0508	0.2254
	β	1.0557	0.0557	0.0400	0.0431	0.2076
	λ	0.7796	0.0796	0.0861	0.0924	0.3040
300	α	1.2457	0.0457	0.0374	0.0394	0.1986
	β	1.0459	0.0459	0.0300	0.0321	0.1792
	λ	0.7657	0.0657	0.0660	0.0703	0.2651
1000	α	1.2155	0.0155	0.0096	0.0099	0.0994
	β	1.0145	0.0145	0.0067	0.0069	0.0832
	λ	0.7219	0.0219	0.0148	0.0152	0.1235

Table 4: Simulation results for the EHL_{ST} distribution

$\left(\alpha = 1.5, \beta = 1.2, \lambda = 1.0\right)$

n	Par	ME	AbsBias	Var	MSE	RMSE
30	α	1.8705	0.3705	0.5020	0.6393	0.7996
	β	1.4449	0.2449	0.7459	0.8059	0.8977
	λ	1.6491	0.6491	1.7722	2.1936	1.4811
50	α	1.7647	0.2647	0.3445	0.4146	0.6439
	β	1.3006	0.1006	0.4233	0.4334	0.6584
	λ	1.4718	0.4718	1.1511	1.3737	1.1721
150	α	1.6985	0.1085	0.1159	0.1276	0.3573
	β	1.1082	0.0918	0.0925	0.1009	0.3176
	λ	1.1875	0.1875	0.3046	0.3397	0.5828
250	α	1.5712	0.0712	0.0634	0.0684	0.2616
	β	1.0660	0.1340	0.0435	0.0615	0.2479
	λ	1.1183	0.1183	0.1494	0.1634	0.4042
300	α	1.5626	0.0626	0.0518	0.0558	0.2362
	β	1.0567	0.1433	0.0346	0.0551	0.2348
	λ	1.1019	0.1019	0.1201	0.1305	0.3612
1000	α	1.5326	0.0326	0.0145	0.0156	0.1249
	β	1.0263	0.1737	0.0084	0.0386	0.1964
	λ	1.0504	0.0504	0.0298	0.0323	0.1798

Table 5: Simulation results for the EHL_{ST} distribution

$(\alpha = 2.5, \beta = 1.0, \lambda = 1.7)$						
n	**Par**	**ME**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	α	2.8609	0.3609	0.7223	0.8525	0.9233
	β	1.3901	0.3901	0.8025	0.9547	0.9771
	λ	2.4215	0.7215	3.5534	4.0739	2.0184
50	α	2.7297	0.2297	0.4341	0.4869	0.6978
	β	1.2328	0.2328	0.3801	0.4342	0.6590
	λ	2.1701	0.4701	1.9601	2.1811	1.4769
150	α	2.5746	0.0746	0.1306	0.1362	0.3690
	β	1.0664	0.0664	0.0600	0.0644	0.2538
	λ	1.8471	0.1471	0.4214	0.4430	0.6656
250	α	2.5432	0.0432	0.0753	0.0771	0.2777
	β	1.0372	0.0372	0.0307	0.0321	0.1792
	λ	1.7845	0.0845	0.2273	0.2345	0.4842
300	α	2.5383	0.0383	0.0629	0.0644	0.2538
	β	1.0322	0.0322	0.0251	0.0261	0.1616
	λ	1.7733	0.0733	0.1846	0.1899	0.4358
1000	α	2.5121	0.0121	0.0188	0.0190	0.1378
	β	1.0098	0.0098	0.0066	0.0067	0.0818
	λ	1.7239	0.0239	0.0502	0.0508	0.2254

exponentiated generalized Pareto (EXGP) distribution [27], Weibull Pareto (WEP) distribution [28], exponentiated Weibull (EXW) distribution [29], exponentiated Burr xii (EBX) distribution [30], Gompertz Lomax (GOLO) distribution [31].

Figure 5 presents the histogram for the Nigeria inflation rates dataset and the corresponding normal Q-Q, total test time (TTT), and hazard rate plots. The histogram shows a high frequency of low inflation rates and a low frequency of high inflation rates in Nigeria. The curve in the TTT plot gives a positive insight that the EHL_{ST} distribution is appropriate for handling such a dataset, given the hazard rate function monotonic feature.

The goodness-of-fit measures include the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) and Hannan-Quinn information criterion (HQIC), are used for model's comparison. We also consider the negative log-likelihood (-LL), Anderson Darling (ANDA), Cramer-von Mises (CVM), Kolmogorov-Smirnov (K-S) statistic and its p-value. The distribution is of a good fit if all the goodness-of-fit results are smaller and the p-value is larger.

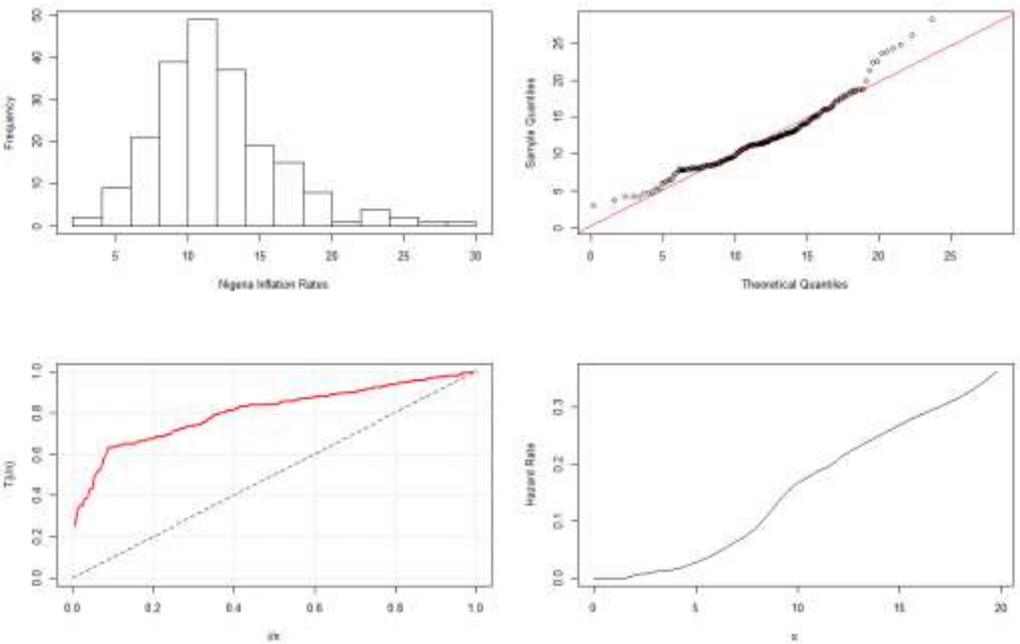

Figure 5: Histogram *(upper left)*, Q-Q plot *(upper right)*, TTT plot *(lower left)* and Hazard rate plot *(lower right)*.

Table 7: MLEs and SEs of the distribution parameters.

DISTRIBUTION	Parameter Estimates			
EHLST $\langle \alpha, \beta, \lambda \rangle$	4.414	143.603	406.095	-
	(0.962)	(94.829)	(174.637)	
TIHLST $\langle \alpha, \lambda \rangle$	0.704	92.053	-	-
	(0.055)	(17.336)		
HLST $\langle \lambda \rangle$	172.708	-	-	-
	(20.486)			
ST $\langle \lambda \rangle$	273.566	-	-	-
	(34.406)			
EHLL $\langle \alpha, \beta, a, b \rangle$	4.405	9.906	4.307	0.017
	(2.772)	(1.580)	(2.711)	(0.004)
EGST $\langle u, v, \lambda \rangle$	3.860	188.632	323.854	-
	(0.654)	(84.587)	114.235	
EXF $\langle a, \lambda, \alpha \rangle$	69.320	0.774	39.769	-
	(28.127)	(0.102)	(21.380)	
EXGP $\langle a, b \rangle$	462.983	105.227	-	-
	(156.931)	(72.340)		
WEP $\langle c, \beta, \theta \rangle$	6.268	0.463	1.499	-
	(1.336)	(0.092)	(0.634)	
EXW $\langle \alpha, \gamma, \lambda \rangle$	4.244	1.568	0.131	-
	(1.737)	(0.263)	(0.027)	
EBX $\langle \alpha, \beta, \theta \rangle$	1.420	190.745	1.636	-
	(0.300)	(49.670)	(0.379)	
GOLO $\langle \theta, \gamma, a, b \rangle$	0.0066	4.847	0.678	0.480
	(0.0020)	(1.393)	(0.200)	(0.106)

Table 8: Goodness-of-fit measures.

Model	-LL	AIC	CAIC	BIC	HQIC	CVM	ANDA	K-S	p-value
EHL_{ST}	586.57	1179.14	1179.26	1189.15	1183.19	0.133	0.899	0.07	0.291
TIHLST	782.47	1568.95	1569.01	1575.63	1571.65	0.136	0.928	0.45	2.2e-16
HLST	792.79	1587.58	1587.60	1590.92	1588.93	0.139	0.915	0.52	2.2e-16
ST	862.55	1727.10	1727.12	1730.43	1728.45	0.152	0.952	0.63	2.2e-16
EHLL	589.93	1187.87	1187.06	1201.22	1193.27	0.208	1.411	0.08	0.173
EGST	587.25	1180.49	1180.61	1190.51	1184.54	0.139	0.941	0.072	0.234
EXF	587.06	1180.13	1180.25	1190.14	1184.18	0.161	1.079	0.075	0.197
EXGP	1282.72	2569.44	2569.50	2576.11	2572.14	0.146	0.972	0.938	2.2e-16
WEP	587.83	1181.66	1181.78	1191.68	1185.71	0.199	1.282	0.08	0.124
EXW	586.83	1179.26	1179.37	1189.27	1183.30	0.148	0.970	0.07	0.276
EBX	623.73	1253.46	1253.58	1263.47	1257.51	0.824	5.335	0.13	0.0006
GOLO	599.93	1207.85	1208.05	1221.20	1213.25	0.479	2.876	0.12	0.0076

The MLEs, standard error (SE), and the goodness-of-fit measures of the fitted distributions are arranged in Tables 7 and 8, respectively. Based on the numerical results in Table 8, the EHL_{ST} distribution is of best fit, with the smallest -LL, AIC, CAIC, BIC, HQIC, CVM, ANDA and largest p-value. The estimated density and distribution functions fits are depicted in Figure 4. Figure 6 shows that the EHL_{ST} distribution fit seems more acceptable in comparison to the other competing distributions; the peaked form of the histogram is well captured given the leptokurtic nature of the EHL_{ST} distribution. Lastly, Table 9 presents the confidence intervals for the EHL_{ST} parameters.

Table 9: Confidence intervals for the EHL_{ST} parameters.

CI	α
95%	$[2.2849 \quad 4.3311]$
	β
	$[20.3281 \quad 112.6559]$
	λ
	$[77.0821 \quad 396.276]$

6.0 Conclusion

This article presents a new three-parameter distribution known as the exponentiated half logistic skew-t distribution using the exponentiated half logistic family of distributions. The flexibility of the skew-t distribution is improved using this family of distributions. This mixture representation is important in deriving several structural properties of new distribution such as the ordinary and incomplete moments, quantile function, entropy, characteristic function and order statistics. The new distribution parameter estimates are derived using the maximum likelihood estimation (MLE) procedure and simulation study

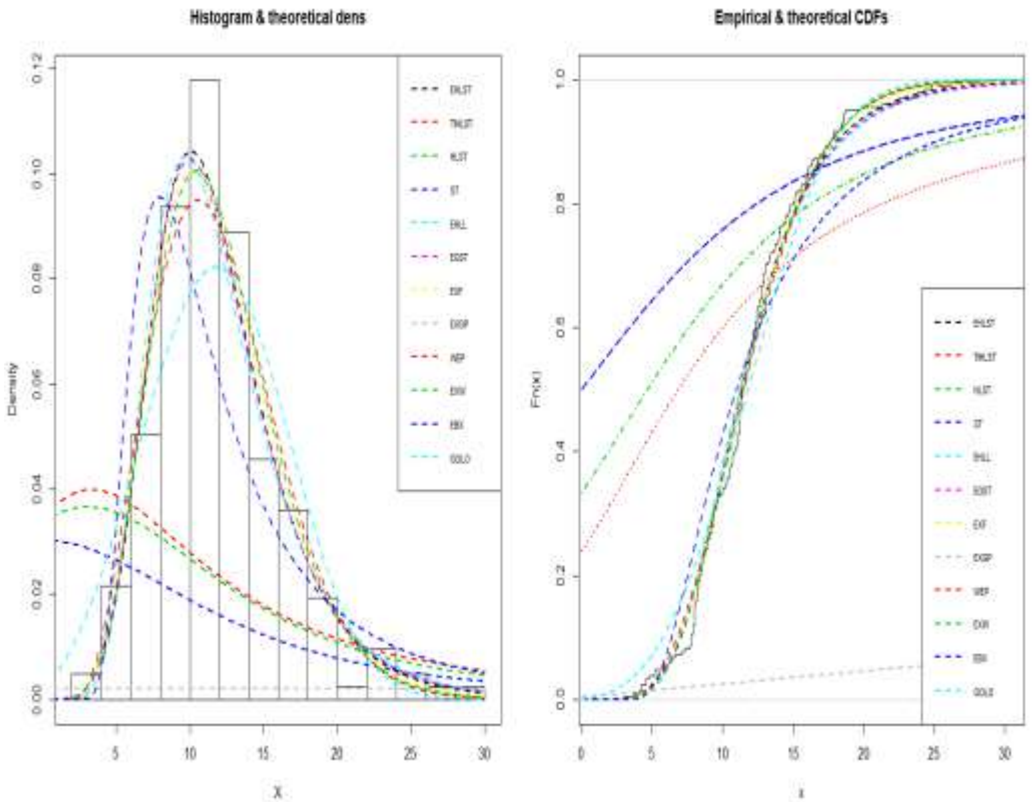

Figure 6: Fitted distributions comparison plots.

showed that the MLE performed well in estimating the parameters of the new distribution. The application using the Nigeria financial dataset indicates that the exponentiated half logistic skew-t distribution outperformed eleven competing distributions. We conclude that the EHL_{ST} distribution is a flexible model when modelling skewed and heavy tail dataset, and should attract wider applications in modelling such datasets. Future research study will compare the performance of the exponentiated half logistic skew-t as an innovation distribution to existing innovation distributions in modeling and predicting volatility using GJR-GARCH framework.

Conflicts of interest

Authors declare that there are no conflicts of interest.

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References

[1] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). Continuous

Univariate Distributions, vol. 2, 2nd edn. New York: Wiley.
[2] Sahu, S. K., Dey, D. K. and Branco, M. D. (2003). A new class of multivariate skew distributions with applications to Bayesian regression models. The Canadian Journal of Statistics, 31, 129–150.
[3] Azzalini, A. and Capitanio, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew-t distribution. J. Roy. Statist. Soc., B, 65, 367-389.
[4] Jones, M. C. (2001). A skew t distribution. In Probability and Statistical Models with Applications (eds C. A. Charalambides, M. V. Koutras and N. Balakrishnan), London: Chapman and Hall, pp. 269–277.
[5] Jones, M. C. (2004). Families of distributions of order statistics. Statistical Methodology, 6, 70-91.
[6] Dikko, H. G and Agboola, S. (2017). Exponentiated Skew-t distribution. Transactions of the Nigerian Association of Mathematical Physics, 4, 251-260.
[7] Adubisi O. D., Abdulkadir, A. and Chiroma, H. (2021). A two-parameter odd exponentiated skew-t distribution with J-shaped hazard rate function. Journal of Statistical Modeling and Analytics. 1(1): 26-46.
[8] Shafiei, S. and Doostparast, M. (2014). Balakrishnan skew-t distribution and associated statistical characteristics. Comm. Statist. Theory Methods, 43, 4109-4122.
[9] Khamis, K. S., Basalamah, D., Ning, W., and Gupta, A. (2017). The Kumaraswamy Skew-t Distribution and Its Related Properties. Communications in Statistics - Simulation and Computation, DOI: 10.1080/03610918.2017.1346801
[10] Aas, K., and Haff, I. H. (2006). The Generalised Hyperbolic Skew Student's t-distribution. Journal of Financial Econometric, 4(2): 275-309.
[11] Shittu, O. I., Adepoju, K. A. and Adeniji, O. E. (2014). On the Beta Skew-t distribution in modelling stock return in Nigeria. International Journal of Modern Mathematical Sciences. 11 (2): 94-102.
[12] Adubisi O. D., Abdulkadir, A., Chiroma, H. and U. F. Abbas (2021). The Type I Half Logistic Skew-t Distribution: A Heavy-Tail Model with Inverted Bathtub Shaped Hazard Rate. Asian Journal of Probability and Statistics. 14(4): 21-40.
[13] Basalamah, D., Ning W. and Gupta, A. (2018). The beta skew-t distribution and its properties. Journal of Statistical theory and practice, 12, 837-860.
[14] Jones, M. C. and Faddy, M. J. (2003). A skew extension of the t distribution, with applications. J. Roy. Statist. Soc., Ser. B., 65 (2): 159–174.
[15] Jones, M. C. (2002). Student's simplest distribution. The Statistician, 51(1): 41-49.
[16] Cordeiro, G. M., Alizadeh, M. and Ortega, E. M. M. (2014). The exponentiated half-logistic family of distributions: Properties and Applications, Journal of Probability and Statistics. Article ID 864396.
[17] Asmussen, S. R. (2003). "Steady-State Properties of GI/G/1". Applied Probability and Queues. Stochastic Modelling and Applied Probability. 51, 266–301.

[18] Kenney, J. F. and Keeping, E. (1962). Mathematics of Statistics. D. Van Nostr and Company.
[19] Moor, J. J. (1988). A quantile alternative for Kurtosis. The Statistician, 37, 25-32.
[20] David, H. A. (1981). Order statistics, John Wiley & Sons, New York.
[21] Rényi, A. (1961). Proceeding of the fourth Berkeley symposium on mathematical statistics and probabilities. First Edition, University of California Press Berkeley.
[22] Tsallis C. (1988). Possible generalisation of Boltzmann-Gibb's statistics. Journal of statistical Physics. 52 (1-2): 479-487.
[23] ------------ (2022). Half-logistic skew-t distribution: Properties and Application (unpublished).
[24] Jamal, F., Reyad, H. M., Ahmed, S. O., Shah, M. A. A. & Altun, E. (2019). Exponentiated half-logistic Lomax distribution with properties and application. Ned University Journal of research. XVI. (2), 1-11.
[25] Dikko, H. G. and Agboola, S. (2018). Statistical properties of the exponentiated generalised skew-t distribution. Journal of the Nigerian Association of Mathematical Physics. 42, 219-228.
[26] Nadarajah S. and Kotz S. (2003). The exponentiated Fréchet distribution. Interstat Electron. J. 14, 1-7.
[27] Lee S. Y. and Kim J. H. T. (2019). Exponentiated Generalised Pareto distribution: Properties and applications towards extreme value theory. Communication in Statistics-Theory and methods. 48 (8), 2014-2038.
[28] Alzaatreh A. and Famoye F. (2013). Weibull-Pareto distribution and its applications. Communication in Statistics-Theory and methods. 42 (9), 1673-1691.
[29] Pal M., Ali M. M. and Woo J. (2006). Exponentiated Weibull distribution. Statistica. 66 (2), 139-147.
[30] Kumar D., Saran J. and Jain N. (2017). The exponentiated burr xii distribution: moments and estimation based on lower record values. Sri Lankan Journal of Applied Statistics. 18 (1), 1-18.
[31] Oguntunde P. E., Khaleel M. A., Ahmed M. T., Adejumo A. O. and Odetunmibi O. A. (2017). New generalisation of the Lomax distribution with increasing, decreasing and constant failure rate. Modeling and Simulation in Engineering. 2017, 1-6.