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The Comparative Study of Gompertz Exponential Distribution and other three Parameter Distributions of Exponential Class

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***Abstract*:** Statistical distributions are very useful in describing and predicting real world phenomena. In this paper, a new continuous model called Gompertz exponential distribution is defined and studied. Its resulting densities and statistical properties were carefully derived and the method of maximum likelihood was proposed in estimating the model parameters. A simulation on R was done to assess the performance of the parameters of the new model. Gompertz exponential distribution was illustrated with an application to real life data. The result shows that Gompertz exponential distribution performs better than other three-parameter distributions such as Kumaraswamy–exponential distribution, Generalized Gompertz distribution and Three-Parameter Lindley distribution.
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Keywords: MLE, GOMPERTZ-exponential, replication, Log Likelihood and AIC

1.0 Introduction

Statistical distributions and their properties are used in modeling naturally occurring phenomena. A large number of distributions have been defined and studied in the literature, which are found to be applicable in the real life. The normal distribution addresses real-valued variables that tend to cluster at a single mean value. The Poisson distribution models discrete rare events. [1] studied the Gompertz distribution and calculated the moment generating function in terms of incomplete or complete gamma functions, and their results are either approximate or left in an integral form. The development of new compound distributions that are more flexible than existing distributions is an important new trend in distribution theory. For instance, the beta-Gompertz distribution [2] and the generalized Gompertz distribution (GGD) [3] were both introduced to take care of skewed data; while the Exponentiated Generalized Weibul-Gompertz distribution and the Gompertz- Lomax distribution (which extends the Lomax distribution using the Gompertz family of distributions) were introduced to take care of non- normal data [4].

The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability. The exponentiated Gompertz distribution defined and studied by [5] is generated from Gompertz random variable by raising the cdf of the Gompertz distribution to a parameter θ.

The current paper focuses on extending the Exponential distribution using the Gompertz family of distributions. The method of Maximum Likelihood Estimator (MLE) for estimating the parameters of the distribution is proposed. A simulation method is used to assess the performance of the parameters of the Gompertz exponential distribution, and an application to real life data sets was provided to assess the potentials of the newly derived distribution. The results obtained are compared with those from Generalized Gompertz distribution, and the proposed distribution is shown to have better performance.

To start with, the cumulative distribution function (cdf) and probability density function (pdf) of the exponential distribution with parameter λ are given by

$$G(x) = 1 - e^{-\lambda x}, \qquad \lambda > 0$$

$$g(x) = \lambda e^{-\lambda x}, \qquad \lambda > 0$$

respectively where λ is referred to as the rate parameter.

According to [6] the cdf and pdf of the Gompertz generalized family of distributions are given by

$$F(x) = 1 - e^{\left(\frac{\theta}{\gamma}\right)\{1-[1-G(x)]^{-\gamma}\}} \quad ; \theta > 0, \gamma > 0$$

$$f(x) = \theta g(x)[1-G(x)]^{-\gamma-1} e^{\left(\frac{\theta}{\gamma}\right)\{1-[1-G(x)]^{-\gamma}\}} \quad ; \theta > 0, \gamma > 0$$

Where θ and γ are additional shape parameters whose role is to vary tail weights. $G(x)$ and $g(x)$ are the cdf and pdf of the parent (or baseline) distribution respectively. The pdf of Gompertz-Exponential distribution is derived by inserting the densities in (1) and (2) into (4)

$$f(x) = \theta\lambda e^{\lambda x\gamma} e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]}, \qquad x > 0, \theta > 0, \gamma > 0, \lambda > 0$$

where θ and γ are scale parameters while λ is rate parameter.

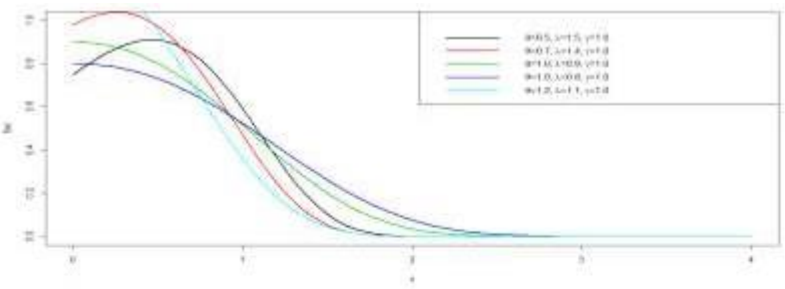

Figure 1: Graph of pdf of GoEp distribution at various parameters.

The cdf of Gompertz-Exponential distribution is derived by inserting the density in (1) into (3),

$$F(x) = 1 - e^{\left(\frac{\theta}{\gamma}\right)\left[\,1-e^{\lambda x \gamma}\right]} \quad ,\theta > 0\,,\gamma > 0\,,\lambda > 0 \qquad (6)$$

where θ and γ are scale parameters while λ is rate parameter.

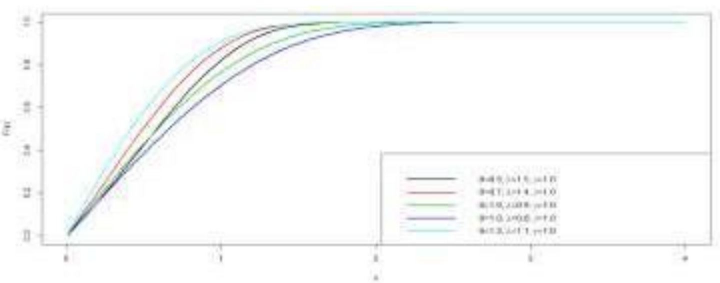

Figure 2: Graph of CDF of GoEp distribution at various parameters.

The Mean and Variance of Gompertz Exponential distribution is obtained thus:

$$E(X) = \int_0^{\infty} x f(x) dx$$

Using (5) for $f(x)$, we obtain

$$E(x) = \int_0^{\infty} x\theta\lambda e^{\lambda x\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left[\,1-e^{\lambda x\gamma}\right]} dx$$

Let $t = \lambda\gamma, p = \theta\lambda \; and \; z = \frac{\theta}{\gamma}$

$$E(x) = \int_0^\infty xpe^z \, e^{tx} e^{-ze^{tx}} \, dx$$

Let $s = pe^z$

$$E(x) = \int_0^\infty xs \, e^{tx} e^{-ze^{tx}} \, dx \qquad (7)$$

Let u$= ze^{tx}$, therefore, $e^{tx} = \frac{u}{z}$

$$du = zte^{tx} dx \,, \quad dx = \frac{du}{zte^{tx}} = \frac{du}{tu}$$

When $x = 0, u = ze^{tx} = ze^{t(0)} = ze^0 = z$

When $x = \infty, u = ze^{tx} = ze^{t(\infty)} = ze^\infty = \infty$

From (7), $E(x) = \int_0^\infty xs \, e^{tx} e^{-ze^{tx}} \, dx$

$$E(x) = \int_z^\infty xs \frac{u}{z} e^{-u} \frac{du}{tu}$$

$$E(x) = \int_z^\infty \frac{s}{zt} xe^{-u} du = \frac{s}{zt} \int_z^\infty xe^{-u} \, du$$

But u $= ze^{tx}$, $e^{tx} = \frac{u}{z}$, $tx = ln\frac{u}{z}$, $x = \frac{1}{t} ln\frac{u}{z}$

Recall that E $(x) = \frac{s}{zt} \int_z^\infty xe^{-u} \, du$; therefore, $E(x) = \frac{s}{zt} \int_z^\infty \frac{1}{t} ln \frac{u}{z} e^{-u} du$

$$E(x) = \frac{s}{zt^2} \int_z^\infty ln \frac{u}{z} e^{-u} du$$

$$E(x) = \frac{s}{zt^2} \int_z^\infty ln \frac{u}{z} e^{-u} du = \frac{pe^z}{\frac{\theta}{\gamma}\lambda^2 \gamma^2} \int_z^\infty ln \frac{u}{z} e^{-u} du = \frac{\theta\lambda e^{\theta/\gamma}}{\frac{\theta}{\gamma}\lambda^2 \gamma^2} \int_z^\infty ln \frac{u}{z} e^{-u} du = \frac{e^{\theta/\gamma}}{\lambda\gamma} \int_z^\infty ln \frac{u}{z} e^{-u} du$$

The mean of GOMPERTZ exponential distribution is hereby derived in a integral form as given below,

$$E(x) = \frac{e^{\frac{\theta}{\gamma}}}{\lambda\gamma} \int_z^\infty ln \frac{u}{z} e^{-u} du$$

$where \;\; z = \frac{\theta}{\gamma}, t = \lambda\gamma, u = ze^{tx} = \frac{\theta}{\gamma}e^{\lambda\gamma x}$

$Recall, Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_0^\infty x^2 f(x)dx$

$f(x) = \theta\lambda e^{\lambda x\gamma} e^{(\frac{\theta}{\gamma})[1-e^{\lambda x\gamma}]}$

$E(x^2) = \int_0^\infty x^2 \theta\lambda e^{\lambda x\gamma} e^{(\frac{\theta}{\gamma})[1-e^{\lambda x\gamma}]} dx$

Let $t = \lambda\gamma, p = \theta\lambda \; and \; z = \frac{\theta}{\gamma}$

$E(x^2) = \int_0^\infty x^2 pe^{tx} e^{z[1-e^{tx}]} dx$

$E(x^2) = \int_0^\infty x^2 pe^{tx} e^{z} e^{-ze^{tx}}$

$E(x^2) = \int_0^\infty x^2 pe^{z} e^{tx} e^{-ze^{tx}} dx$

Let $s = pe^z$

$E(x^2) = \int_0^\infty x^2 s \, e^{tx} e^{-ze^{tx}} dx$ (8)

Let u $= ze^{tx}$, therefore, $e^{tx} = \frac{u}{z}$

$du = zte^{tx}dx$, $dx = \frac{du}{zte^{tx}} = \frac{du}{tu}$

When $x = 0, u = ze^{tx} = ze^{t(0)} = ze^0 = z$

When $x = \infty, u = ze^{tx} = ze^{t(\infty)} = ze^\infty = \infty$

From (8), $E(x^2) = \int_0^\infty x^2 s \; e^{tx} e^{-ze^{tx}} dx$

$E(x^2) = \int_z^\infty x^2 s \frac{u}{z} e^{-u} \frac{du}{tu}$

$$E(x^2) = \int_z^{\infty} \frac{s}{zt} x^2 e^{-u} du = \frac{s}{zt} \int_z^{\infty} x^2 e^{-u}\, du$$

But u$= ze^{tx}$, $e^{tx} = \frac{u}{z}$, $tx = ln\frac{u}{z}$, $x = \frac{1}{t} ln\frac{u}{z}$, $x^2 = (\frac{1}{t} ln\frac{u}{z})^2 = \frac{1}{t^2}\ (ln\frac{u}{z})^2$

$$E(x^2) = \frac{s}{zt^3} \int_z^{\infty} (ln\frac{u}{z})^2 e^{-u} du = \frac{pe^z}{\frac{\theta}{\gamma}\lambda^3\gamma^3} \int_z^{\infty} (ln\frac{u}{z})^2 e^{-u} du$$

$$= \frac{\theta\lambda e^{\frac{\theta}{\gamma}}}{\frac{\theta}{\gamma}\lambda^3\gamma^3} \int_z^{\infty} (ln\frac{u}{z})^2 e^{-u} du$$

$$E(x^2) = \frac{e^{\frac{\theta}{\gamma}}}{\lambda^2\gamma^2} \int_z^{\infty} (ln\frac{u}{z})^2 e^{-u} du$$

The variance of GOMPERTZ exponential distribution is hereby derived in a integral form as

$$Var(x) = \frac{e^{\frac{\theta}{\gamma}}}{\lambda^2\gamma^2} \int_z^{\infty} (ln\frac{u}{z})^2 e^{-u} du - \left(\left(\frac{e^{\frac{\theta}{\gamma}}}{\lambda\gamma} \int_z^{\infty} ln\frac{u}{z} e^{-u} du \right)^2 \right)$$

$$where\ \ z = \frac{\theta}{\gamma}\ , t = \lambda\gamma, u = ze^{tx} = \frac{\theta}{\gamma} e^{\lambda\gamma x}$$

The parameters of the GoEp distribution can be estimated using the method of Maximum Likelihood Estimation (MLE) as follows: let $x_1, x_2, \ldots, x_n$ denote random samples each having the pdf of the GOMPERTZ-exponential distribution, then the likelihood function is given by

$$f(x_1, x_2, \ldots, x_n; \theta, \gamma, \lambda) = \prod_{i=1}^{n} \{\theta\lambda e^{\lambda x_i \gamma} e^{(\frac{\theta}{\gamma})[1-e^{\lambda x_i \gamma}]}\} \qquad (9)$$

Let l denote the log-likelihood function, that is, let $l = log f(x_1, x_2, \ldots, x_n; \theta, \gamma, \lambda)$, then

$$l = nlog\theta + nlog\lambda + \lambda\gamma \sum_{i=1}^{n} x_i + \frac{\theta}{\gamma} \sum_{i=1}^{n} [1 - e^{\lambda x_i \gamma}]$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\gamma} \sum_{i=1}^{n} [1 - e^{\lambda x_i \gamma}]$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \gamma \sum_{i=1}^{n} x_i - \theta \sum_{i=1}^{n} x_i e^{\lambda x_i \gamma}$$

$$\frac{\partial l}{\partial \gamma} = \lambda \sum_{i=1}^{n} x_i - \frac{\theta}{\gamma^2} - \lambda \sum_{i=1}^{n} x_i e^{\lambda x_i \gamma}$$

Solving $\frac{dl}{d\theta} = 0$, $\frac{dl}{d\lambda} = 0\ and\ \frac{dl}{d\gamma} = 0$ simultaneously gives the maximum likely

estimates of parameters θ, λ *and* γ. Meanwhile, the solution cannot be got analytically except numerically when data sets are available. Soft-wares like R, MATLAB, MAPLE and so on could be used to get the estimates.
The expressions for the reliability function, hazard function (or failure rate), reversed hazard function and odd functions are all derived and established below.
The expressions for the reliability function, hazard function (or failure rate), reversed hazard function and odd functions are all derived and established below.
Reliability or survival function can be obtained from

$$S(x) = 1 - F(x) \tag{10}$$

Therefore, the reliability function of the GoEp distribution is given by

$$S(x) = 1 - \{1 - e^{(\frac{\theta}{\gamma})[1-e^{\lambda x \gamma}]}\}$$

$$S(x) = e^{(\frac{\theta}{\gamma})[1-e^{\lambda x \gamma}]} \quad , x > 0,\ \theta > 0\ , \gamma > 0\ , \lambda > 0 \tag{11}$$

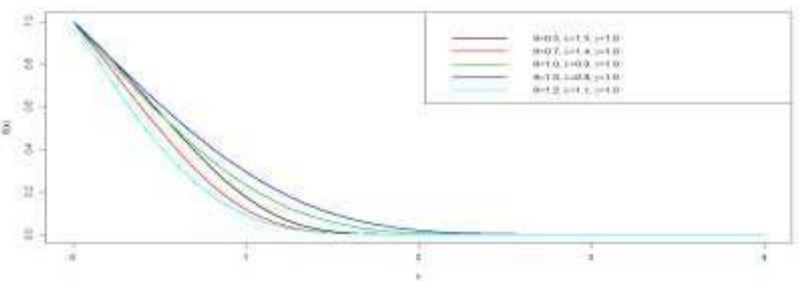

Figure 3: Graph of survival function of GoEp distribution at various parameters.

Hazard function can be obtained from

$$h(x) = \frac{f(x)}{S(x)} \tag{12}$$

Therefore the hazard function of GoEp distribution is given by

$$h(x) = \frac{\theta \lambda e^{\lambda x \gamma} e^{(\frac{\theta}{\gamma})[1-e^{\lambda x \gamma}]}}{\left\{e^{(\frac{\theta}{\gamma})[1-e^{\lambda x \gamma}]}\right\}}$$

$$h(x) = \theta \lambda e^{\lambda x \gamma} \qquad x > 0\ , \theta > 0\ , \gamma > 0\ , \lambda > 0 \tag{13}$$

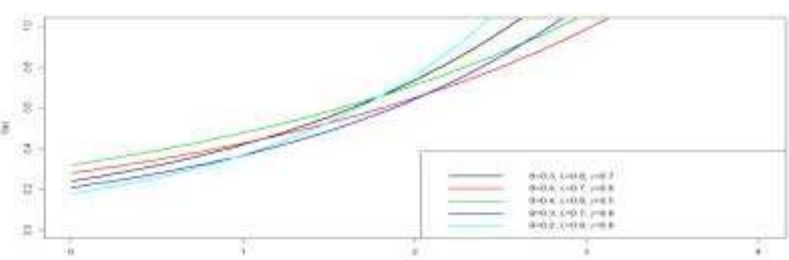

Figure 4: Graph of Hazard function of GoEp distribution at various parameters.

The reversed hazard function can be derived from

$$r(x) = \frac{f(x)}{F(x)} \qquad (14)$$

Therefore, the reversed hazard function of GoEp distribution is given by

$$r(x) = \frac{\left\{\theta\lambda e^{\lambda x\gamma} e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]}\right\}}{\left\{1 - e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]}\right\}} \;, \theta > 0\,, \gamma > 0\,, \lambda > 0 \qquad (15)$$

Odds function can be derived from $O(x) = \frac{F(x)}{S(x)}$ (16)

Therefore, the odds function for the GoEp distribution is given by

$$O(x) = \frac{1 - e^{\left(\frac{\theta}{\gamma}\right)\{1-[1-G(x)]^{-\gamma}\}}}{e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]}} \;, \theta > 0\,, \gamma > 0\,, \lambda > 0 \qquad (17)$$

Quantile function can be derived from $Q(u) = F^{-1}(u)$ (18)

Therefore, the quantile function of the GoEp distribution is derived as follows:

$$F(x) = 1 - e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]} \;, \quad \theta > 0\,, \gamma > 0\,, \lambda > 0$$

Let $F(x) = u$

Therefore, $u = 1 - e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]}$

$$e^{\left(\frac{\theta}{\gamma}\right)[1-e^{\lambda x\gamma}]} = 1 - u$$

$$\left(\frac{\theta}{\gamma}\right)\left[1 - e^{\lambda x\gamma}\right] = \log(1-u)$$

$$\left[1 - e^{\lambda x\gamma}\right] = \left(\frac{\gamma}{\theta}\right)\log(1-u)$$

$$e^{\lambda x\gamma} = 1 - \left(\frac{\gamma}{\theta}\right)\log(1-u)$$

$$\lambda x\gamma = log[1 - \left(\frac{\gamma}{\theta}\right)\log(1-u)]$$

$$x = \frac{1}{\lambda\gamma} log[1 - \left(\frac{\gamma}{\theta}\right)\log(1-u)]$$

Therefore, $Q(u) = \frac{1}{\lambda\gamma} log\left[1 - \left(\frac{\gamma}{\theta}\right)\log(1-u)\right]$ (19)

Where $u \sim Uniform\ (0,1)$.

Random numbers can be generated from the GoEp distribution using

$$x = \frac{1}{\lambda\gamma} log\left[1 - \left(\frac{\gamma}{\theta}\right)\log(1-u)\right] \qquad (20)$$

The median of the GoEp distribution can be derived by substituting $u = 0.5$ in (19) as follows:

$$Median = \frac{1}{\lambda\gamma} log\left[1 - \left(\frac{\gamma}{\theta}\right)\log(1-0.5)\right]$$

$$Median = \frac{1}{\lambda\gamma} log\left[1 - \left(\frac{\gamma}{\theta}\right)\log 0.5\right] \quad (21)$$

Other quantiles can also be derived from (19) by substituting the appropriate values of *u*.

2.0 Materials and Methods

We conducted simulation study for the purpose of investigating the behaviour of the parameters of the new model. Data sets were generated from the GoEp distribution using R software for simulation with a replication m=1000 and random sample of sizes 50, 100 and 150. The simulation was conducted for three different cases by varying the parameter values. The selected parameter values are $\theta = 0.5$, $\lambda = 0.5$, and $\gamma = 0.5$; and $\theta = 1$, $\lambda = 1$, and $\gamma = 1$; and $\theta = 2$, $\lambda = 2$, and $\gamma = 2$ respectively. Results of the simulation study are shown in Tables 1, 2 and 3

A first real life data was on the strengths of 1.5cm glass fibres of workers at the UK national Physical Laboratory. The data had previously been used by [7], [8], [9] and [4] while the second data was on the lifetimes of 50 devices. The data was drawn from [10] and was used by [3].

3.0 Results

Tables 1, 2 and 3, are the Root Mean Square Error (RMSE) which reduces for the selected parameter values as the sample size increases. This indicates that the parameters of Gompertz Exponential distribution are stable. Table 4 revealed that both data I and data II are negatively skewed with coefficients of skewness -0.8999 and –

0. 1378 respectively. The performance ratings of the Gompertz-exponential are shown in Tables 5 and 6 below

Table 5 and 6 are the distribution that has the lowest –LL and AIC. The smaller the values of the –LL and AIC the better the fit of the data and this is judged to be the best out of the competing distributions. With this regard to this information, the competing distributions are ranked in the order of best to the least. This implies that Gompertz exponential distribution has the smallest –LL and AIC compared to other distributions like Kumaraswamy Exponential distribution, Generalized Gompertz distribution and Three-Parameter Lindley distribution

4.0 Discussion of Findings

It can be deduced from Tables 1, 2 and 3 that the Root Mean Square Error (RMSE) reduces for all the selected parameter values as the sample size increases. This gives credence to the stability of the parameters of Gompertz Exponential distribution. The Table 4 is about result of the skewness of the real life data I and II which were negatively skewed.

Tables 5 and 6 give information about the fits of the distributions that were compared. The distribution with the lowest –LL and AIC is judged to be the best fit out of the competing distributions. The competing distributions can be ranked in the following order of best to the least such as from Gompertz-

Exponential distribution, Kumaraswamy-Exponential distribution, Generalized Gompertz distribution and three–parameter Lindley distribution for the data on the strengths of glass fibres. The result shows that the data on the lifetimes of 50 devices, the competing distributions can be ranked in the following order (best to the least): GOMPERTZ-exponential distribution, Generalized Gompertz distribution, Kumaraswamy-Exponential distribution and three–parameter Lindley distribution.

5.0 Conclusion

The simulation study conducted showed that the parameters of the Gompertz Exponential distribution are stable. Also the values for biasedness that were generated were small, indicating that the maximum likelihood estimates of the GOMPERTZ exponential distribution are not too far from the true parameter values. The absolute bias and the root mean square values decreased as the sample size increased. An application to real life data shows that the GOMPERTZ exponential distribution performs better than its competitors.

6.0 References

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Table 1: Simulation study at $\theta = 0.5$, $\lambda = 0.5$ and $\gamma = 0.5$

n	Parameters	Means	Bias	Std Error	RMSE
50	$\theta = 0.5$	0.4076	0.0924	3.5553	0.2667
	$\lambda = 0.5$	0.7414	-0.2414	6.4635	0.3595
	$\gamma = 0.5$	0.2177	0.2823	1.8996	0.1949
100	$\theta = 0.5$	0.6787	- 0. 1787	2.9989	0.1732
	$\lambda = 0.5$	3.4324	-2.9324	7.4332	0.2726
	$\gamma = 0.5$	0.7455	-0.2455	3.2873	0.1813
150	$\theta = 0.5$	0.6169	-0.1169	3.7880	0.1589
	$\lambda = 0.5$	0.5608	-0.0608	3.4433	0.1515
	$\gamma = 0.5$	0.3438	0.1562	2.1128	0.1187

Table 2: Simulation study at $\theta = 1, \lambda = 1$ and $\gamma = 1$

n	Parameters	Means	Bias	Std Error	RMSE
	$\theta = 1$	1.2657	-0.2657	5.1286	0.3203
50	$\lambda = 1$	0.6676	0.3324	2.6972	0.2323
	$\gamma = 1$	1.5581	-0.5581	6.2999	0.3550
	$\theta = 1$	1.4071	-0.4071	3.9013	0.1975
100	$\lambda = 1$	0.6378	0.3622	1.7612	0.1327
	$\gamma = 1$	1.9934	-0.9934	5.5040	0.2346
	$\theta = 1$	1.7204	-0.7204	2.3016	0.1239
150	$\lambda = 1$	0.4463	0.5537	0.5930	0.0629
	$\gamma = 1$	3.0393	-2.0393	4.0560	0.1644

Table 3: Simulation study at $\theta = 2, \lambda = 2$ and $\gamma = 2$

n	Parameters	Means	Bias	Std Error	RMSE
	$\theta = 2$	0.7456	1.2544	2.0670	0.2033
50	$\lambda = 2$	5.3744	-3.3744	14.7060	0.5423
	$\gamma = 2$	1.3379	0.6621	3.6434	0.2699
	$\theta = 2$	1.1800	0.8200	0.9210	0.1709
100	$\lambda = 2$	2.6480	-0.6480	6.4870	0.2547
	$\gamma = 2$	2.0600	-0.0600	5.0120	0.2239
	$\theta = 2$	1.9950	0.0050	2.6570	0.1331
150	$\lambda = 2$	2.0090	-0.0090	2.6540	0.1330
	$\gamma = 2$	2.1890	-0.1890	2.9070	0.1392

Table 4: The descriptive statistics for the two datasets above are provided in the table below.

Parameters	N	Min.	Q_1	Median	Q_3	Mean	Max.	Skewness	Kurtosis	Variance
Dataset 1	63	0.550	1.375	1.590	1.685	1.507	2.240	- 0.8999	3.9238	0.1051
Dataset II	50	0.10	13.50	48.50	81.25	45.69	86.00	- 0.1378	1.4139	1078.15

Table 5 Performance of Compared Distributions using data on 1.5cm glass fibres

Distributions	Estimates	-LL	AIC
Gompertz-Exponential	$\theta = 0.605800$ $\lambda = 0.145560$ $\gamma = 25.05831$	14.8081	35.61621
Kumaraswamy-Exponential	$\theta = 1756$ $\lambda = 7.001$ $\gamma = 0.2599$	15.91371	37.82742
Generalized Gompertz	$\theta = 0.49258$ $\lambda = 0.48005$ $\gamma = 0.54581$	83.20132	172.4026
Three-Parameter Lindley	$\theta = 0.499886$ $\lambda = 0.002048$ $\gamma = 0.01903$	102.9163	211.8325

Table 6: Performance of Compared Distributions using data on Lifetimes of 50 devices

Distributions	Estimates	-LL	AIC
Gompertz-Exponential	$\theta = 0.01050$ $\lambda = 0.92555$ $\gamma = 0.02193$	235.3308	476.6617
Generalized Gompertz	$\theta = 0.00143$ $\lambda = 0.044$ $\gamma = 0.2599$	235.3920	476.7840
Kumaraswamy Exponential	$\theta = 0.12631$ $\lambda = 0.46598$ $\gamma = 0.15839$	238.4378	482.8756
Three-Parameter Lindley	$\theta = 0.0049523$ $\lambda = 0.0103508$ $\gamma = 0.0001513$	322.0525	650.105