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An Application of Second Derivative Ten Step Blended Block Linear Multistep Methods for the Solutions of the Holling Tanner Model and Van Der Pol Equations.

Omagwu Samson[1]* & Joshua Kyaharnan Victor[2]

[1]Department of Mathematics & Statistics, Kaduna Polytechnics, Kaduna, Nigeria;
[2]Department of Mathematics, University of Jos, Jos, Nigeria
*samsonomagwu@yahoo.com

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***Abstract*:** This paper is concerned with the accuracy and efficiency of the ten step blended block linear multistep method for the approximate solution of Holling Tanner Model and the Van Da Pol equations. The main methods were derived by blending of two linear multistep methods using continuous collocation approach. These methods are of uniform order eleven. The stability analysis of the block methods indicates that the methods are A–stable, consistent and zero stable hence convergent. Numerical results obtained using the proposed new block methods were compared with those obtained by the well-known ODE solver ODE15 S to illustrate its accuracy and effectiveness. The proposed block methods are found to be efficient and accurate hence recommended for the solution of stiff initial value problems.
Keywords: A-Stable, Blended Block, Continuous Collocation, Linear Multistep Methods, Stiff ODEs

Introduction

Mathematical modeling of many problems in real life, Science, Medicine, Engineering and the like gave rise to systems of linear and nonlinear Differential Equations. In some cases, the differential equations could be solved analytically while in other case like the Holling Tanner equations and the Van Der Pol equations they are too complicated to be solved by analytical methods. Thus solving such problems becomes an uphill task hence the application of numerical methods for approximate solutions to these differential equations.

The Holling Tanner model is a pre-predator model which was developed independently in the early twentieth century by “Lotka [1]” an American

Biologist and "Voltera [2]" an Italian mathematician. Holling Tanner model commonly called Lotka-Voltera Equations are given in the form

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy - dxy \qquad (1)$$

Where x and y are the number of prey and predators respectively and a, b, c, and d are positive constants.. Equation (1) was modified by "Roenzweng and MacArthur [3]" to give (2) as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{mx}{a+x}y$$
$$\frac{dy}{dt} = \delta y + \gamma \frac{mx}{a+x}y \qquad (2)$$

The addition of the functional response term $P(x) = \frac{mx}{a+x}$ (3)

Was "suggested by Holling [4]", he explained that the functional response should not be a monotonic increasing function but a bounded function. "Further modification to (1) was made by Tanner [5]" to give as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{my}{D+x}$$
$$\frac{dy}{dt} = sy\left(1 - \frac{hy}{k}\right) \qquad (4)$$

This system of non linear differential equations was referred to as the Holling-Tanner dimensionless population model, where r, k, m, D, s and h are positive constants with h representing the number of prey required to support one predator at equilibrium and t the dimensionless time variable systems.

The Van der Pol's Equation is an important kind of second-order non-linear auto-oscillatory equation. It is a non-conservative oscillator with non-linear damping.

$$y_1' = y_2$$
$$y_2' = -y_1 - \mu y_2(1 - y_1^2)$$
(5)

$$\mu = 500, y_1(0) = 2, y_2(0) = 0$$
$$0 \le x \le 40, \quad h = 0.1$$

In this paper, the application of the ten step blended block linear multistep method for the numerical solutions of the Holling Tanner and the Van Der Pol's equations (2) and (5) respectively were considered. A potentially good numerical method for the solution of stiff system of ordinary differential equations (ODEs) must have good accuracy and some wide region of absolute stability "as was discussed by Enright [6]". One of the first and most important stability requirements for linear multistep methods is A-stability "as was proposed by Enright [7]". The ten step blended block linear multistep methods is of a high order and A stable hence the application of the method here which makes it suitable for the solution of non linear ODEs.

The solution of Holling Tanner has been considered by "Collom [8]" where a block hybrid Adams Moulton Method was used and by "Kumleng [9]" where Generalized Adams methods were used. Many discussed the solution of linear and nonlinear ODEs from different basis functions, among them are Onumanyi [10], Butcher [11], Gamal [12], Ezzeddine13], Kumleng[14], Sahi [15] and so on.

The Ten Step Blended Linear Multistep Method

The ten step blended linear multistep method is constructed based on the continuous finite difference approximation approach using the interpolation and collocation criteria described by Lie and Norset [16] called multistep collocation (MC) and block multistep methods "by Onumanyi [10, 17]". We define based on the interpolation and collocation methods the continuous form of the k- step 2nd derivative new method as

$$\alpha_{k-1}(x)=\sum_{i=0}^{t+m-1}\alpha_{j,i+1}x^{i}\quad j=0,1,\ldots,t-1$$

$$\beta_{j}(x)=\sum_{i=0}^{t+m-1}\beta_{j,i+1}x^{i},\ j=0,1,2,\ldots,m-1$$

and $$\lambda_{k}(x)=\sum_{i=0}^{t+m-1}\lambda_{k,i+1}x^{i},\ j=0,1,2,\ldots,m-1$$

$$y(x)=\sum_{j=1}^{1}\alpha_{j}(x)y_{n+j}+h\sum_{j=0}^{m-1}\beta_{j}(x)f_{n+j}+h^{2}\lambda_{k}(x)y''_{n+k}\qquad(6)$$

Points, h is the step size and "from Onumanyi [10]", we obtain our matrices D $C=D^{-1}$ by the imposed conditions expressed as DC = I

Where:

$$D=\begin{bmatrix}
1 & x_n & x_n^2 & \ldots & x_n^{t+m-1}\\
1 & x_{n+1} & x_{n+1}^2 & \ldots & x_{n+1}^{t+m-1}\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
1 & x_{n+k-1} & x_{n+k-1}^2 & \cdots & x_{n+k-1}^{t+m-1}\\
0 & 1 & 2\bar{x}_0 & \ldots & (t+m-1)\bar{x}_0^{\,t+m-2}\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 1 & 2\bar{x}_{m-1} & \ldots & (t+m-1)\bar{x}_{m-1}^{\,t+m-2}\\
0 & 0 & 2 & \cdots & (t+m-2)(t+m-1)\bar{x}_0^{\,t+m-3}\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & 2 & \ldots & (t+m-2)(t+m-1)\bar{x}_{m-1}^{\,t+m-3}
\end{bmatrix}\qquad(7)$$

$$C=\begin{pmatrix}
\alpha_{01} & \alpha_{11} & \ldots & \alpha_{t-1,1} & h\beta_{01} & h\lambda_{01} & \ldots & h\lambda_{m-1,1}\\
\alpha_{02} & \alpha_{12} & \ldots & \alpha_{t-1,2} & h\beta_{02} & h\lambda_{02} & \ldots & h\lambda_{m-1,2}\\
. & . & \ldots & . & . & . & . & .\\
. & . & \ldots & . & . & . & . & .\\
. & . & \ldots & . & . & . & . & .\\
\alpha_{0,t+m} & \alpha_{1,t+m} & \ldots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & h\lambda_{0,t+m} & \ldots & h\lambda_{m-1,t+m}
\end{pmatrix}=D^{-1}\qquad(8)$$

respectively.

In this case, k=10,t=1 and m=12 and it continuous form expressed in the form of (6) is

$$y(x) = \alpha_9(x) y_{n+9} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} + h^2 \lambda_{10}(x) y''_{n+10} \tag{9}$$

Using the approach of [17]. The matrix form of

$$D = \begin{bmatrix} 1 & (x_n+9h) & (x_n+9h)^2 & \cdots & (x_n+9h)^{12} \\ 0 & 1 & 2x_n & \cdots & 6x_n^{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2(x_n+5h) & \cdots & 6(x_n+5h)^{11} \\ 0 & 0 & 2 & \cdots & 30(x_n+5h)^{10} \end{bmatrix} \tag{10}$$

(3.44) Using the Maple software, the inverse of the matrix in (10) is obtained and its elements are used in obtaining the continuous coefficients and substituting these continuous coefficients into (9) yields the continuous form of our new method. The continuous form as:

$$\begin{aligned}
&+\left(\frac{105t^2}{2h} + \frac{7241t^3}{72h^2} - \frac{1524587t^4}{17280h^3} + \frac{730667t^5}{16200h^4} - \frac{4578533t^6}{311040h^5} + \frac{193247t^7}{60480h^6} - \frac{12917t^8}{27648h^7} + \frac{1769t^9}{38880h^8} - \frac{163t^{10}}{57600h^9} + \frac{29t^{11}}{285120h^{10}} - \frac{\tau^{12}}{622080h^{11}} + \frac{10066113h}{19711200}\right) f_{n+6} \\
&+\left(\frac{270t^2}{7h} - \frac{149t^3}{2h^2} + \frac{222011t^4}{3360h^3} - \frac{95839t^5}{2800h^4} + \frac{2057947t^6}{181440h^5} - \frac{50473t^7}{20160h^6} + \frac{20033t^8}{53760h^7} - \frac{479t^9}{12960h^8} + \frac{23t^{10}}{50400h^9} - \frac{19t^{11}}{221760h^{10}} + \frac{\tau^{12}}{725760h^{11}} - \frac{1072359h}{197120}\right) f_{n+7} \\
&+\left(\frac{405t^2}{16h} + \frac{11019t^3}{224h^2} - \frac{394133t^4}{8960h^3} + \frac{73661t^5}{3200h^4} - \frac{3735073t^6}{483840h^5} + \frac{34823t^7}{20160h^6} - \frac{24053t^8}{92160h^7} + \frac{9547t^9}{362880h^8} - \frac{1367t^{10}}{806400h^9} - \frac{t^{11}}{15840h^{10}} - \frac{\tau^{12}}{967680h^{11}} + \frac{7429239h}{3942400}\right) f_{n+8} \\
&+\left(\frac{4889t^2}{504h} - \frac{36007309t^3}{1905120h^2} - \frac{861584837t^4}{50803200h^3} + \frac{2037630739t^5}{228614400h^4} + \frac{329678017t^6}{109734912h^5} - \frac{8244667t^7}{12192768h^6} - \frac{35826937t^8}{348364800h^7} - \frac{2864549t^9}{274337280h^8} + \frac{10333t^{10}}{15240960h^9} - \frac{51203t^{11}}{2011806720h^{10}} + \frac{4609\tau^{12}}{10973491200h^{11}} + \frac{176321121h}{110387200}\right) f_{n+9} \\
&\left(\frac{9t^2}{20h} - \frac{7409t^3}{8400h^2} - \frac{321691t^4}{403200h^3} - \frac{14233t^5}{33600h^4} + \frac{629521t^6}{4354560h^5} - \frac{317t^7}{9600h^6} + \frac{7063t^8}{1382400h^7} - \frac{t^9}{1890h^8} + \frac{17t^{10}}{483840h^9} - \frac{t^{11}}{739200h^{10}} + \frac{\tau^{12}}{43545600h^{11}} - \frac{42039h}{788480}\right) f_{n+10} \\
&+\left(5t^2 + \frac{7381t^3}{756h} - \frac{177133t^4}{20160h^2} - \frac{84095t^5}{18144h^3} - \frac{341693t^6}{217728h^4} + \frac{8591t^7}{24192h^5} - \frac{7513t^8}{138240h^6} + \frac{605t^9}{108864h^8} - \frac{11t^{10}}{30240h^8} + \frac{t^{11}}{72576h^9} - \frac{\tau^{12}}{4354560h^{10}} - \frac{4671h^2}{7168}\right) g_{n+10}
\end{aligned} \tag{11}$$

Gives the ten discrete methods which constitute the ten step blended block linear multistep method

$$y_{n+1} = y_{n+9} + \frac{2368}{467775}hf_n - \frac{154228}{467775}hf_{n+1} - \frac{31936}{22275}hf_{n+2} - \frac{53888}{155925}hf_{n+3} - \frac{32896}{17325}hf_{n+4} + \frac{848}{155925}hf_{n+5} - \frac{32896}{17325}hf_{n+6} - \frac{53888}{155925}hf_{n+7} - \frac{3196}{22275}hf_{n+8} - \frac{154228}{467775}hf_{n+9} + \frac{2368}{467775}hf_{n+10}$$

$$y_{n+2} = y_{n+9} - \frac{825601}{1231718400}hf_n + \frac{7269199}{547430400}hf_{n+1} - \frac{17157049}{45619200}hf_{n+2} - \frac{1743469}{1368576}hf_{n+3} - \frac{16439647}{22809600}hf_{n+4} - \frac{57626401}{45619200}hf_{n+5} - \frac{54653963}{68428800}hf_{n+6} - \frac{4523543}{3801600}hf_{n+7} - \frac{8595923}{9123840}hf_{n+8} - \frac{2540383643}{4926873600}hf_{n+9} - \frac{453103}{136857600}hf_{n+10} - \frac{117943}{1244160}h^2y''_{n+10}$$

$$y_{n+3} = y_{n+9} - \frac{19}{138600}hf_n + \frac{7269199}{547430400}hf_{n+1} - \frac{57}{24640}hf_{n+2} + \frac{2403}{107800}hf_{n+3} - \frac{9297}{7700}hf_{n+4} - \frac{25407}{30800}hf_{n+5} - \frac{1767}{1540}hf_{n+6} - \frac{3411}{3850}hf_{n+7} - \frac{16713}{15400}hf_{n+8} + \frac{3587609}{7761600}hf_{n+9} - \frac{3}{3080}hf_{n+10} - \frac{19}{280}h^2y''_{n+10}$$

$$y_{n+4} = y_{n+9} - \frac{3175}{38320128}hf_n + \frac{180625}{15380512}hf_{n+1} - \frac{748625}{89413632}hf_{n+2} + \frac{140375}{3193344}hf_{n+3} - \frac{325945}{709632}hf_{n+4} - \frac{14194175}{12773376}hf_{n+5} - \frac{679375}{709632}hf_{n+6} - \frac{3200875}{3193344}hf_{n+7} - \frac{1856375}{1824768}hf_{n+8} - \frac{522610955}{1072963584}hf_{n+9} - \frac{77425}{38320128}hf_{n+10} - \frac{575}{7168}h^2y''_{n+10}$$

$$y_{n+5} = y_{n+9} - \frac{435569}{958003200}hf_n - \frac{21860567}{3832012800}hf_{n+1} + \frac{14825963}{447068160}hf_{n+2} - \frac{9471361}{79833600}hf_{n+3} + \frac{5199599}{17740800}hf_{n+4} - \frac{170469863}{319334400}hf_{n+5} + \frac{13395017}{17740800}hf_{n+6} - \frac{14221079}{15966720}hf_{n+7} + \frac{342968359}{3193364400}hf_{n+8} + \frac{4776367597}{26824089600}hf_{n+9} + \frac{4671}{7168}hf_{n+10} - \frac{575}{7168}h^2y''_{n+10}$$

$$y_{n+6} = y_{n+9} - \frac{1693}{35481600}hf_n - \frac{9843}{15769600}hf_{n+1} - \frac{106137}{27596800}hf_{n+2} + \frac{14799}{985600}hf_{n+3} - \frac{84681}{1971200}hf_{n+4} + \frac{417609}{15769600}hf_{n+5} - \frac{1066431}{1971200}hf_{n+6} - \frac{1016451}{985600}hf_{n+7} - \frac{3992841}{3942400}hf_{n+8} - \frac{484449641}{993484800}hf_{n+9} - \frac{7827}{3942400}hf_{n+10} - \frac{575}{7168}h^2y''_{n+10}$$

$$y_{n+7} = y_{n+9} + \frac{17}{3742200}hf_n - \frac{2119}{29937600}hf_{n+1} + \frac{4579}{8731800}hf_{n+2} - \frac{31}{12474}hf_{n+3} - \frac{599}{69300}hf_{n+4} - \frac{61519}{2494800}hf_{n+5} + \frac{4757}{69600}hf_{n+6} - \frac{150463}{311850}hf_{n+7} - \frac{276181}{249480}hf_{n+8} - \frac{96395491}{209563200}hf_{n+9} - \frac{4141}{3742200}hf_{n+10} + \frac{19}{280}h^2y''_{n+10}$$

$$y_{n+8} = y_{n+9} - \frac{460423}{8622028800}hf_n + \frac{522197}{766402560}hf_{n+1} - \frac{184633}{45619200}hf_{n+2} + \frac{3569021}{239500800}hf_{n+3} - \frac{6120217}{159667200}hf_{n+4} + \frac{23693849}{31934400}hf_{n+5} - \frac{11141161}{239500800}hf_{n+6} + \frac{4522543}{26611200}hf_{n+7} - \frac{26007199}{45619200}hf_{n+8} - \frac{18217146173}{34488115200}hf_{n+9} - \frac{516149}{191600640}hf_{n+10} + \frac{117943}{1244160}h^2y''_{n+10}$$

$$y_{n+10} = y_{n+9} + \frac{435569}{958003200}hf_n - \frac{21860567}{3832012800}hf_{n+1} + \frac{14825963}{447068160}hf_{n+2} - \frac{9471361}{79833600}hf_{n+3} + \frac{5199599}{17740800}hf_{n+4} - \frac{170469863}{319334400}hf_{n+5} + \frac{13395017}{17740800}hf_{n+6} - \frac{14221079}{15966720}hf_{n+7} + \frac{342968359}{3193364400}hf_{n+8} + \frac{4776367597}{26824089600}hf_{n+9} + \frac{41198923}{191600640}hf_{n+10} + \frac{4671}{7168}h^2y''_{n+10}$$

Stability Analysis of the New Methods

In this section, we consider the analysis of the newly constructed methods. Their convergence is determined and their regions of absolute stability plotted.

3.1 Convergence The convergence of the new block methods is determined BBLMM we use the "approach of Ehigie [20]" for linear multiste methods where he expressed the methods in the matrix form as shown below. Following the work of Ehigie and Okunuga [21], we observed that the seven step block method is zero stable as the roots of the equation $det\left(r\left(A - Cz - D1z^2\right) - B\right) = 0$ are less than or equal to 1. Since the block using the "approach by Fatunla [18]" and " Chollom [19]" for linear multistep methods, where the block methods are represented in a single block, r point multistep method of the form **Zero Stability of the BBLMM for k=10.** To determine the zero stability of the method is consistent and zero-stable, the method is convergent "as discused by Henrici [22]". These new methods are consistent since their orders are 11, it is also zero-stable, above all, there are A – stable as can be seen in figure 1. The new ten step discrete methods that constitute the block method have the following orders and error constants as shown below:

$$(11,11,,11,11,11,11,11,11,11,11)^T$$

The ten step blended block multistep methods has uniform order of and error constants of $C_{13} = \left(\frac{-2925}{117056}, \frac{-125}{145609}, \frac{-315}{131715320}, \frac{685}{5200526}, \frac{-251}{405260}, \frac{185}{973120}, \frac{-1775}{67757060}, \frac{448}{30518703}, \frac{-224}{117802196}, \frac{-725}{1705600}\right)^T$

Regions of Absolute Stability of the Methods

The absolute stability regions of the newly constructed blended block linear multistep methods (8) and (12) are plotted using [20] by reformulating the methods into a characteristic equation of the form

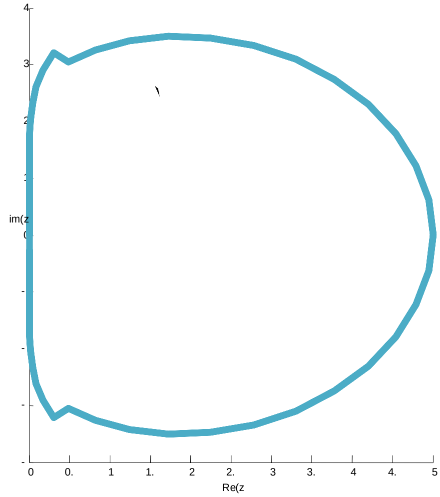


Figure 1: Absolute Stability Region For BBLMM For K=10.

This absolute stability region is A – stable since it consist of the set of points in the complex plane outside the enclosed figure.

Numerical Experiment

We report here a numerical example on stiff problem taken from the literature using the solution curve. In comparison, we also report the performance of the new blended block linear multistep methods and the well-known Matlab stiff ODE solver ODE15S on the same problems and on the same axes.

Problem 1 Holling Tanner Predator-Prey Equations

The Holling Tanner Predictor – prey model is expressed mathematically by the following ivp

$$y_1' = \sigma y_1\left(1-\frac{y_1}{K}\right)-\frac{My_1y_2}{D+y_1}$$

$$y_2' = y_2\left(1-\frac{hy_2}{y_1}\right)$$

$D=0.2, r=1, \sigma=0.02, k=0.2, h=1.2, M=0.02$

$$y_1' = y_1\left(1-\frac{y_1}{0.2}\right)-\frac{0.02y_2}{0.2+y_1}$$

$$y_2' = y_2\left(1-\frac{1.2y_2}{y_1}\right)$$

$$y_1(0)=0.1, y_2(0)=0.1$$

$$0 \le x \le 100, \quad h=0.1$$

This Model describes the dynamics of a generalize predator which feeds on a prey.

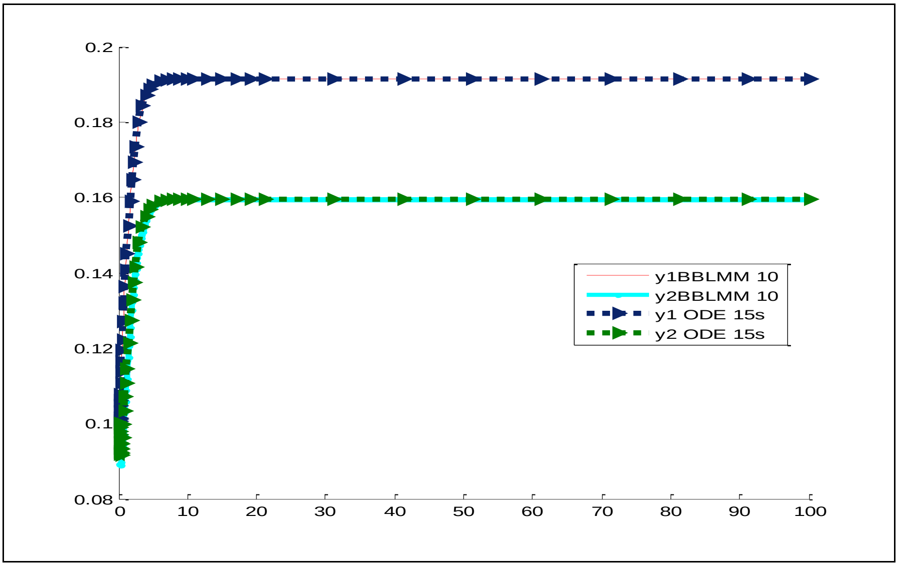
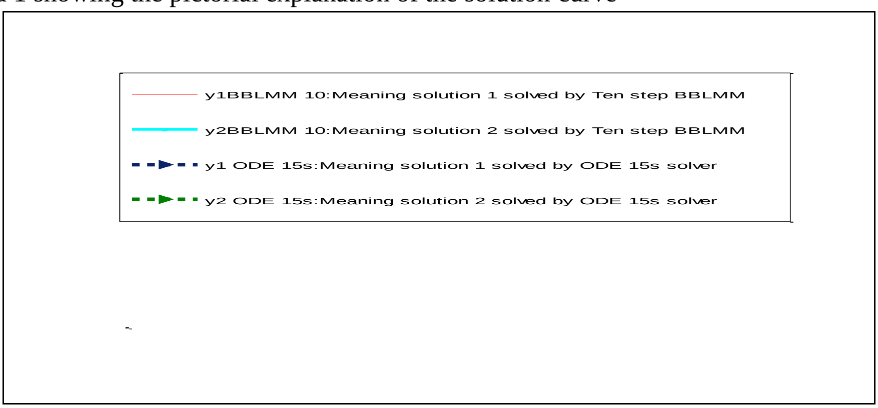


Figure 2: Solution Curve of the Problem 1 Computed by Ten Step BBLMM

Legend 1 showing the pictorial explanation of the solution curve



Problem 2: Van der pol's Equations

$$y_1' = y_2$$
$$y_2' = -y_1 - \mu y_2(1 - y_1^2)$$
$$\mu = 40, y_1(0) = 2, y_2(0) = 0 \qquad 0 \le x \le 40, \quad h = 0.1$$

The Van der Pol's Equation is an important kind of second-order non-linear auto-oscillatory equation. It is a non-conservative oscillator with non-linear damping.

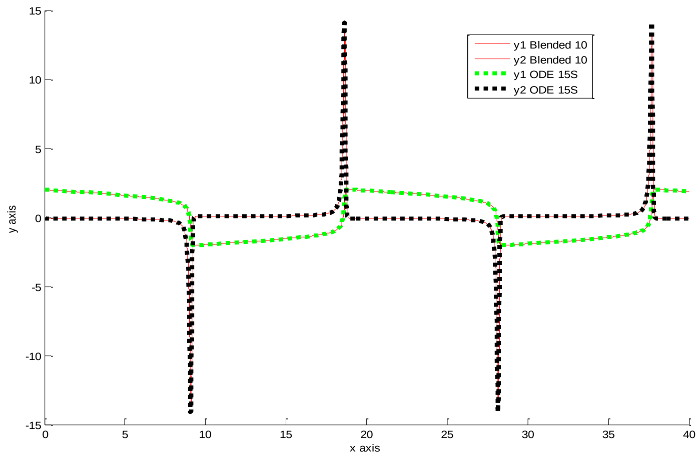
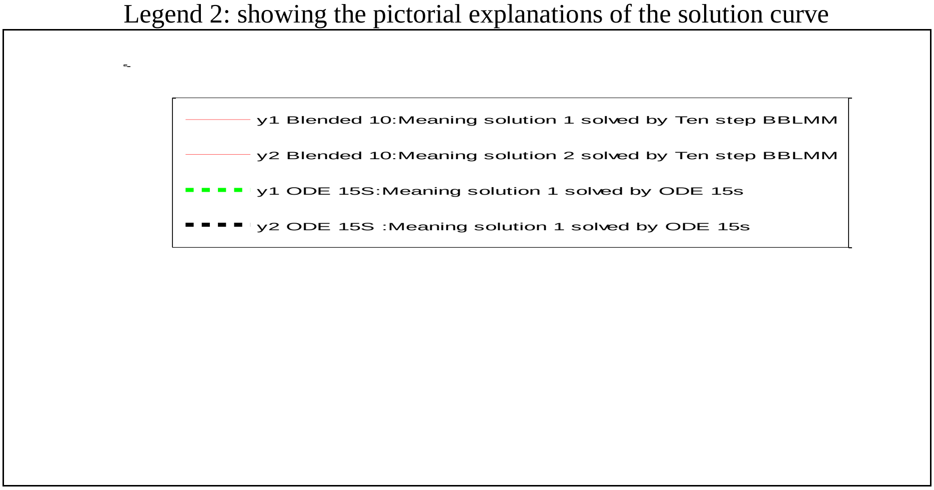


Figure 3: Solution Curve Of The Problem 2 Computed By Ten Step BBLMM

Legend 2: showing the pictorial explanations of the solution curve



Discussion and Conclusion

Problem 1 which is a predator-prey model shows that the predator grows logistically with intrinsic growth rates and carrying capacity proportional to the size of the prey. The solution curves reveal that the curve of our BBLMM and that of the ODE 15s solver are on each other which implies that coexistence of both the prey and the predator provides a stabilizing influence. Our solution is at par with ODE 15s.

Van der Pol's equation in problem 2 is a non-conservative oscillator with non linear damping energy dissipated at high amplitude. From the solution curves, the legend shows that the trajectories trace the motion of a single point through a flow with a limit circle where the trajectories spiral into or away from the limit circle. Our solution curves compete favourably with ODE 15s solver.

It can be seen clearly from the curves that our new methods compete favourably with the well known ODE 15s for the problems solved in problem 1and 2. It was also observed that the new methods have better stability regions than the conventional Adams Moulton method for step number 10.

Recommendations

These methods are recommended for the solutions of stiff system of ODEs since they are A-stable which implies a wider range of stability for effective performance.

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