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On a Truncated Accelerated Plan for Two Component Parallel Systems under Ramp-Stress Testing Using Masked Data for Weibull Distribution

Braimah Joseph Odunayo

Department of Mathematics and Statistics,
Faculty of Physical Sciences,
Ambrose Alli University,
Ekpoma, Edo State, Nigeria
Ojbraimah2014@gmail.com

***Abstract*:** Several studies on design of Acceptance Life Test (ALT) focused on a subsystem (single system) totally ignoring its internal design. In most cases, it is not always possible to identify the components that cause the system failure or the cause can only be identified by a subset of its component resulting in a masked observation. This paper therefore investigates into the development of ramp-stress accelerated life testing for a high reliability parallel system that consist of two dependent components using masked failure data. This type of testing may be very useful in a twin-engine plane or jet. A ramp-stress results when stress applied on the system increases linearly with time. A parallel system with two dependent components is taken with dependency modeled by G umbel-Hougaard copula. The stress-life relationship is modeled using inverse power law and cumulative exposure model is assumed to model the effect of changing stress. The method of maximum likelihood is thereafter used for estimating design parameters. This optimal plan consists in finding the optimal stress rate using D-optimality criterion by minimizing the reciprocal of the determinant of Fisher information matrix. The projected plan is also explained using a real life example and sensitivity analysis carried out. This formulated model can help guide and assist engineers to obtain reliability estimates quickly with high reliability products that are sustainable.

***Keywords*:** Accelerate, Life test, Ramp-stress, Gumbel-Hougaard copula, Masked data, Fisher information matrix, D-optimality criterion, Dependent components.

Acronyms

ALT- accelerated life test
Avar- asymptotic variance
ML- maximum likelihood
cdf- cumulative distribution function
pdf- probability distribution function

1.0 Introduction

After production process has been carefully controlled up till the finished products, high reliability products of modern times have to be subjected to accelerated life test to detect early failures. This also helps the manufacturer to obtain timely reliability estimates about his products and live on in today's competitive market. Such products may be subject to different stress loading schemes. Such stress schemes include: constant-stress, step-stress, progressive-stress and their various combinations depending upon how they are to be used in service and other limitations both theoretical and practical [1, 2]. A ramp-stress results when stress applied linearly increases with time. A stress can be applied under fully accelerated environmental conditions in which all the test specimens are tested under accelerated condition or partially accelerated environmental conditions where they are tested both under normal and accelerated conditions [3, 4].

Several accelerated life test plans under different stress loading schemes have been devised in some literatures [5, 6]. Nevertheless, both plans are meant for a single system (i.e, a sub-system) with its internal configuration totally ignored. In many cases, it is not always probable to identify the component that caused the system failure or the cause of failure can only be identified by a subset of its component [7]. An observation is said to be masked when event cause of the system failure is not known except that it is as a result of some subset of the component of the system have used the exact maximum likelihood estimation of life time distribution of the component in the series system using masked data [8, 9]. [10] have used the Bayes estimation of component reliability from masked system-life data. [8, 9] have extended the results of [11] to a three component series system of exponential distribution. [12] has used the masked interval data in the series system of exponential components. Formulation of a ramp-stress ALT plan for a parallel system with two dependent components but without masking has been studied by [13]. This paper centered on formulation of a ramp-stress ALT plan for a system with parallel configuration in the presence of masked failure data. Such a testing may prove to be useful in a twin-engine plane or jet. A parallel system with two dependent components is taken with dependency modeled by Gumbel-Hougaard copula. The optimal stress rate is obtained using D-optimality criterion. A numerical example was used to demonstrate application of the developed projected plan and sensitivity analysis was also carried out to examine its robustness.

2.0 The Model

In this section, the model for formulation of a ramp-stress ALT plan for a system with parallel pattern in the presence of masked failure data is developed and its life distribution function with (and) likelihood functions are obtained.

Assumptions

i. The dependency between the two components of the parallel system is modeled by Gumbel-Hougaard copula evaluated at two Weibull survival (reliability) marginals, viz., $\bar{G}_1(.)$ and $\bar{G}_2(.)$ with shape parameter β_1 and β_2 , and common scale parameter θ.η is the measure of association between the two components.

ii. The censoring time τ is pre specified.
iii. The two components of the system cannot fail simultaneously.
iv. Failed parallel systems are not replaced during the test.
v. The occurrence of masking is independent of the failure cause and time.
vi. The effect of changing stress is modeled by the linear cumulative exposure model.
vii. The stress applied to test units is continuously increased at a constant ramp rate ***k*** from zero.
viii. The inverse power law holds for stress-life relationship, i.e,

$$\eta(s(t)) = e^{\mu}\left(\frac{S_0}{S(t)}\right)^{\alpha} \qquad \textbf{(i)}$$

where μ is the characteristics of the product and α is the shape parameter, $s_{(0)}$ is the stress level under normal operating conditions or design stress and $s_{(t)}$ is a linear function of time in ramp-stress at time t.

2.1 Test

The reliability testing procedure is as follows:

i. If *n* independent and identical parallel systems are put to test and their failure times along with the cause of failure are recorded. An observation is said to be masked if its corresponding cause of failure cannot be recorded.
ii. The test is terminated when all the systems fail.

2.2 Parallel System

A parallel system fails if all the components fail. The configuration of a parallel system with two components is shown in Figure 1.

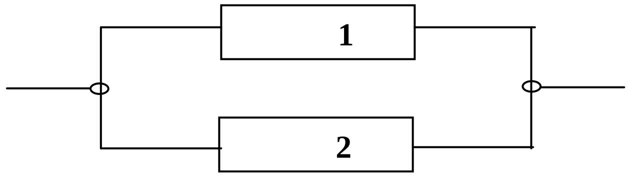

Figure 1: Parallel System

2.3 Copula Function

The dependency existing between the marginal random variables in bivariate and multivariate distributions are described by a copula [1]. The copula describes the way in which the marginals are linked together on the basis of their association.

Suppose X_1 and X_2 are two random variables and let $G_1(x_1)$ and $G_2(x_2)$ be their respective marginal reliability functions. If $H(x_1,x_2)$ are their joint reliability function, thus, according (Therefore according) to Sklar's theorem, there exists a copula reliability function $C(x_1,x_2)$ such that for all that (x_1, x_2) in the defined array:

$$\overline{H}(x_1,x_2) = C\left(\hat{G}_1(x_1), \hat{G}_2(x_2)\right) \text{ (ii)}$$

Amongst the Gumbel-Hougaard copula is defined as:

$$C_{\mu}(a,b) = e^{-\left((-\log_e[a])^{\mu} + (-\log_e[b])^{\mu}\right)^{\frac{1}{\mu}}}$$

(iii)

where $1 \le \mu \le \infty$ characterizes the relationship between the two variables. Gumbel-Hougaard copula is uni-parametric and symmetrical.

2.4 Reliability Function for Bivariate-Weibull Distribution

The reliability function for Bivariate Weibull distribution is obtained by using Weibull reliability marginals in Gumbel-Hougaard reliability function. Using equation (iii) and assumption (i), according to [16], equation (iv) is arrived at:

$$C\left(\bar{G}_1(t_{i1}), \bar{G}_2(t_{i2})\right) = e^{-\left(\left(\frac{t_1}{\mu}\right)\beta_1\alpha + \left(\frac{t_2}{\mu}\right)\beta_2\alpha\right)^{\frac{1}{\alpha}}} \quad \text{(iv)}$$

Where t = testing time, μ = quality parameter, β = risk and α = shape parameter.

2.5 The Bivariate Weibull Reliability Function for Ramp-Stressed Data

The pdf of the bivariate Weibull distribution is given as:

$$f(t,\beta,\alpha) = \frac{\beta+1}{\beta}\alpha e^{-\alpha t\left(1-e^{-\beta\alpha t}\right)} \quad \text{(v)}$$

The Bivariate Weibull reliability function of a parallel system using Gumbel-Hougaard copula as given by [16] is

$$\hat{G}(t_1,t_2) = e^{-\left(\left(\frac{t_1}{\mu}\right)\beta_1\alpha + \left(\frac{t_2}{\mu}\right)\beta_2\alpha\right)^{\frac{1}{\alpha}}} \quad \text{(vi)}$$

The bivariate joint probability density function is given as:

$$g(t_1,t_2) = e^{-\left(\left(\frac{t_1}{\mu_1}\right)\beta_1\alpha + \left(\frac{t_2}{\mu_2}\right)\beta_2\alpha\right)^{\frac{1}{\alpha}}} \beta_1\beta_2\left(\frac{t_1}{\mu_1}\right)\beta_1\alpha\left(\frac{t_2}{\mu_2}\right)\beta_2\alpha \times \frac{\left(\left(\frac{t}{\mu_1}\right)\beta_1\alpha + \left(\frac{t}{\mu_2}\right)\beta_2\alpha\right)^{-2+\frac{1}{\alpha}\left(\left(\frac{t_1}{\mu_1}\right)\beta_1\alpha + \left(\frac{t_2}{\mu_2}\right)\beta_2\alpha\right)^{\frac{1}{\alpha}}+\alpha-1}}{t_1 t_2} \quad \text{(vii)}$$

where $t_1 \geq 0, t_2 \geq 0, \mu_1 > 0, \beta_i > 0, i = 1,2 \; and \; \alpha \geq 1$, μ_i are scale parameters, β_i are shape parameters and α is the association between the two variables. From the linear cumulative model, the joint reliability function of the parallel system under ramp-stress scheme is given as:

$$\hat{F}(t_1,t_2) = \hat{G}\left(E(t_1), E(t_2)\right) \quad \text{(viii)}$$

where $\hat{G}(.,.)$ is the underlying bivariate Weibull reliability function with assumed scale parameter taken to be one (1).

$$E(t) = \int_0^t \frac{1}{\mu(S(a))} da \quad \text{(ix)}$$

Equation above is the cumulative harm (damage) model at t. Therefore, the joint cumulative distribution (reliability) function and joint probability (failure) density function respectively of the system under ramp-stress loading are given as:

$$F(t_1,t_2) = e^{-\left(\left(\int_0^{t_1}\frac{1}{\mu(S(a))}da\right)\beta_1\alpha + \left(\int_0^{t_2}\frac{1}{\mu(S(a))}da\right)\beta_2\alpha\right)^{\frac{1}{\alpha}}} \quad \text{(x)}$$

Therefore,

$$F(t_1,t_2) = e^{-\left(\left(\frac{t_1}{\varphi_1}\right)\beta_1\alpha + \left(\frac{t_2}{\varphi_2}\right)\beta_{22}\alpha\right)^{\frac{1}{\alpha}}} \quad \text{(xi)}$$

$$F(t_1,t_2) = e^{-\left(\left(\frac{t_1}{\varphi_1}\right)\beta_{11}\alpha + \left(\frac{t_2}{\varphi_2}\right)\beta_{22}\alpha\right)^{\frac{1}{\alpha}}} \beta_1\beta_2\left(\frac{t_1}{\varphi_1}\right)\beta_{11}\alpha\left(\frac{t_2}{\varphi_2}\right)\beta_{22}\alpha \quad \text{(xii)}$$

$$\times \frac{\left(\left(\frac{t}{\varphi_1}\right)\beta_1\alpha + \left(\frac{t}{\varphi_2}\right)\beta_2\alpha\right)^{-2+\frac{1}{\alpha}\left(\left(\frac{t_1}{\varphi_1}\right)\beta_{11}\alpha + \left(\frac{t_2}{\varphi_2}\right)\beta_{22}\alpha\right)^{\frac{1}{\alpha}}+\alpha-1}}{t_1 t_2}$$

where

$$\varphi_i = \left(e^{\gamma_0}\left(\frac{S_0}{k}\right)\gamma_{i(1-\gamma_i)}\right)^{\frac{1}{1+\gamma_i}}$$

is the scale parameter, (xiii)

$$\beta_{ii} = \beta_i(1+\gamma_i) \quad \text{(xiv)}$$

2.6 The D-Optimality

The D-optimality criterion is used in minimizing the reciprocal of the determinant of Fisher information matrix, the Fishers smaller value of the determinant corresponds to a higher (joint) precision of the estimators of α, β [14].

2.7 Likelihood Function

This section deals with the case of the complete system but masked data. Likelihood for a parallel system is developed for two dependent components. Suppose we consider a sample of n-systems each consisting of two dependent components in parallel. Suppose T_i is the life time of system I and T_{ij} is the life time of component j in system i, i=1,2......n and j=1,2, then

$$T_i = \max(T_{i1}, T_{i2}) \quad \text{(xv)}$$

The probability that the system fails due to component 1, when $0 \le t_1 < \infty$ is obtained as:

$$P[T_{i2} \le t_i, t_i < T_{i1} \le t_i + \Delta t_i] = F_{T1}(t_i + \Delta t_i) - F_{T1,T2}(t_i, t_i)$$

$$= F_{T1}(t_i) - F_{T1,T2}(t_i, t_i)$$

As $\Delta \to 0$ and since F_{T1} is absolutely differentiable,

$$= 1 - F_{T2}(t_i) - \bar{F}_{T1,T2}(t_i, t_i)$$

$$\bar{F}_{T2}(t_i) - \bar{F}_{T1,T2}(t_i, t_i)$$

$$\text{Therefore, } L \infty - \frac{\partial}{\partial t_1}\left(\bar{F}_{T1,T2}(t_1, t_2)\right) I_{t_1 = t_i, t_2 = t_i} \quad \text{(xvi)}$$

Also, the probability that the system fails due to component 2, when $0 \le t_i \le \infty$ is obtained as: $P[T_{i1} \le t_i, t_i < T_{i2} \le t_i + \Delta t_i] = F_{T2}(t_i + \Delta t_i) - F_{T1,T2}(t_i, t_i)$

$$= F_{T2}(t_i) - F_{T_1,T_2}(t_i, t_i)$$

As $\Delta t_i \to 0$ and since F_{T2} is absolutely differentiable,

$$= 1 - F_{T1}(t_i) - \bar{F}_{T1,T2}(t_i, t_i)$$

$$\bar{F}_{T1}(t_i) - \bar{F}_{T1,T2}(t_i, t_i)$$

$$\text{Therefore, } L \infty - \frac{\partial}{\partial t_2}\left(\bar{F}_{T_1,T_2}(t_1, t_2)\right) \mathrm{I}_{t_1 = t_i, t_2 = t_i} \quad \text{(xvii)}$$

2.8 The log-likelihood (L)

The log-likelihood of an n parallel system is as given below:

$$L = \prod_{S_i=1}^{n_1}\left(-\frac{\partial}{\partial t_1}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \times \prod_{S_i=2}^{n_2}\left(-\frac{\partial}{\partial t_2}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \times \prod_{S_i=1}^{n_1+n_2}\left(-\frac{\partial}{\partial t_1}\left(\bar{F}_{T1,T2}(t_1, t_2)\right) - \frac{\partial}{\partial t_2}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \quad \text{(xviii)}$$

where n is specified by the control engineer (experimenter).

$$L = \sum_{S_i=1}^{n_1}\log\left(-\frac{\partial}{\partial t_1}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \times \sum_{S_i=2}^{n_2}\log\left(-\frac{\partial}{\partial t_2}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \times \sum_{S_i=1}^{n_{12}}\log\left(-\frac{\partial}{\partial t_1}\left(\bar{F}_{T1,T2}(t_1, t_2)\right) - \frac{\partial}{\partial t_2}\left(\bar{F}_{T1,T2}(t_1, t_2)\right)\right) \quad \text{(xix)}$$

3.0 Simulated of Parameter Estimation

The Maximum Likelihood Estimates of ρ_1, ρ_2, β_1 *and* β_2 are obtained using R statistical software. The simulation is carried out following [15].

The algorithm is given below:

Select n units and put them to test.

Specify the masking level (ρ).

iii. Calculate n_{12} such that $n_{12} \approx \left(\frac{\rho}{n} \times 100\right)$.

iv. Arbitrarily select a random sample of size n from the system life time, and the set of component causing the system failure $(t_1,s_1),\ldots,(t_n,s_n)$.

These random samples are generated following the steps below:

i. Generate n_{12} observations using the system cumulative (i.e, product's lifetime) distribution, which is known as time to failure.

ii. Generate n - n_{12} observations using the system cumulative distribution, and determine Si for each i, (i=1, 2,…,n-n_{12}), which gives the set of observations where the cause of system failure is known.

In table 1, the time to failure in minutes and the component that fails during the experiment is as shows below;

Table 1: Simulated data estimates

System No.	Time to Failure (t_i)	Component Failure-cause (S_i)
1.	0.0516	(2)
2.	0.1504	(1,2)
3.	0.1944	(1,2)
4.	1.2604	(1)
5.	3.1649	(1,2)
6.	5.437	(2)
7.	5.5425	(1)
8.	8.5725	(2)
9.	10.0166	(1)
10.	10.9509	(2)

System n umber 1, 2 and 2 has the least time to failure with component (2), (1, 2) and (1, 2) causing the failure respectively while system number 9 and 10 with system number 1 and 2 causing the failure respectively.

Maximum Likelihood Estimates (MLE) of the Design Parameters

The ML estimates of the design parameters obtained using simulated data estimates in table 1 are:

$\rho_1 = -2.2,\ \rho_2 = 0.5, \beta_1 = 0.35\ and\ \beta_2 = 0.24$

In selecting an optimum test plan, there is a need to estimate the design parameters $\rho_{01},\ \rho_{02}, \beta_{11}\ and\ \beta_{12}$.These estimates at times may affect the values of the resulting decision variables significantly. Therefore, their incorrect choice may result in poor estimate of the design constant stress. Therefore, it is significant to carry out a sensitivity analysis to evaluate the robustness of the resulting Acceptance Life Test plan. Sensitivity analysis helps to identify the design parameters $\rho_0, \rho_1, \beta_1\ and\ \beta_2$which need to be estimated with care to avoid the risk of obtaining wrong solutions. An Acceptance Life Test plan is said to be robust if a small departure in any has no effect in relative change in the optimal plan. The percentage deviations (PD) of the optimal settings are obtained as $PD = \left(\frac{|T^{**} - T^{*}|}{T^{*}}\right) \times 100$, where T^{*} is obtained with the given design parameters and T^{**} is obtained when the parameter is miss-specified.

Table 2 illustrates the optimal test plans for various deviations from the design parameter estimates. The results explain that the optimal setting of T is robust to the small variance from baseline parameter estimates.

Table 2: Sensitivity Analysis for changes in design parameters
$\rho_1 = -3.45, \rho_2 = 0.65, S_0 = 20, \beta_1 = 0.35 \; and \; \beta_2 = 0.25, n = 10, n_{12} = 3 \; and \; \alpha = 0.8$

Parameter	%	K	T^{**}	Percent Deviation (%)
β_1	-5%	1.75	0.000574	3.6526
β_1	+5%	1.74	0.000596	7.6891
β_2	-5%	1.78	0.000583	5.3500
β_2	+5%	1.67	0.000587	5.9500
ρ_0	-5%	1.57	0.000585	5.5979
ρ_0	+5%	2.024	0.000585	5.5872
ρ_1	-5%	1.59	0.000589	4.9225
ρ_1	+5%	1.81	0.000581	6.2877

4.0 Discussion

This study deals with optimal planning of accelerated life test of a parallel system with two dependent components under ramp-stress loading for a Weibull distribution. The dependency is modeled by Gumbel-Hougaard copula evaluated at Weibull reliability marginals. The optimal plan consists in finding optimal stress rate using D-optimality criterion. A hypothetical ramp-stress ALT experiment for a parallel system with two dependent components is considered to illustrate the methods described in this paper. From the simulated dataset, system n umber 1, 2 and 2 were found to has the least time to failure with component (2), (1, 2) and (1, 2) causing the failure respectively while system number 9 and 10 with system number 1 and 2 causing the failure respectively.

5.0 Conclusion

This study has carefully developed a ramp-stress Acceptance Life Test for accelerated environmental conditions for a high reliability parallel system consisting of two dependent mechanisms using masked failure data. Such an experiment may be very useful in a two-engine plane or jet. The relationship between the two components is modeled using inverse power law and cumulative exposure. The method of maximum likelihood was used for estimating design parameters. D-optimality criterion was used to find the optimal stress rate using by minimizing the reciprocal of the determinant of Fisher information matrix. Conclusively, a simulation study (using R) is used to illustrate the method developed. The sensitivity analysis results proved that the proposed plan is better for a small departure from baseline parameters.

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