



SOME ASPECTS OF TOPOLOGICAL SORTING

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Abstract: In this paper, we provide an outline of most of the known techniques and principal results pertaining to computing and counting topological sorts, realizers and dimension of a finite partially ordered set, and identify some new directions.

Key words: Partial Ordering, topological sorting, realizer, dimension.

1. Introduction

Topological sorting has been found particularly useful in sorting and scheduling problems such as *PERT charts* used to determine an ordering of tasks, *graphics* to render objects from back to front to obscure hidden surfaces, *painting* when applying paints on a surface with various parts and identifying errors in *DNA* fragment assembly (Knuth, 1973; Skiena, 1997; Rosen, 1999). In this paper, we provide systematically an outline of most of the techniques

and principal results pertaining to computing and counting topological sorts, realizers and dimension of a partially ordered set (Poset), and identify some new directions.

2. Definitions of some terms and related results pertaining to partially ordered sets

We borrow these definitions from various sources (Brualdi *et al.*, 1992; Jung, 1992; Trotter 1991). Let (P, \leq_P) denote a finite partially ordered set along with an implicit assumption that P denotes the

underlying set and \leq_P denotes its order relation. Moreover, \leq stands for reflexive partial order and $<$ for strict partial order.

For an element $a \in P$, the set $U(a) = \{x \in P \mid a <_P x\}$ is called an *open upset* of a . The set $U[a] = \{x \in P \mid a \leq_P x\}$ is called a *closed upset* of a . For any nonempty subset $A \subseteq P$, $U(A) = \{x \in P \mid \exists a \in A, a <_P x\}$ and $U[A] = \{x \in P \mid \exists a \in A, a \leq_P x\} = U(A) \cup A$.

Similarly, the open and closed down sets can be defined. Note that the closed upset and the closed down set are also called *filter* and *ideal* respectively.

An element $y \in P$ is said to cover an element $x \in P$ if $x \neq y$ and $x \leq_P y$ with no element $z \in P$ such that $x \leq_P z \leq_P y$. Sometimes we also say that y is an *immediate successor* of x or x is an *immediate predecessor* of y . For every $x, y \in P$, if $x \leq_P y$ then the pair (x, y) is said to be a *comparability* of P . Two elements $a, b \in P$ are called comparable, denoted $a \perp b$ or $a|b$ if either $a \leq_P b$ or $b \leq_P a$; and incomparable denoted $a||b$, if both $a \not\leq_P b$ and $b \not\leq_P a$. Also, $a <_P b$, iff $a \leq_P b$ and $a \neq b$.

The incidence of a poset (P, \leq_P) , denoted $inc(P)$, is defined as the set $\{(x, y) \in P \times P : x || y \text{ in } \leq_P\}$.

A pair $(x, y) \in inc(P)$ is called a *critical pair* if

$u \leq_P x$ in \leq_P implies $u \leq_P y$ in \leq_P and

$v \geq_P y$ in \leq_P implies $v \geq_P x$ in \leq_P for all $u, v \in P$.

Also, the set of all critical pairs is denoted $crit(P)$.

A subset A of a poset (P, \leq_P) is called a *subposet* if the *suborder* \leq_A defined on A , is the restriction of \leq_P on pairs of elements of A . In other words, a subposet is a subset of (P, \leq_P) with the induced order.

A chain in P is a subposet of P which is a linear order. The length of a chain C of P is $|C| - 1$. An antichain in P is a subset of P containing elements that are mutually incomparable.

Two posets (P, \leq_P) and (Q, \leq_Q) are *isomorphic*, $P \cong Q$, if there exists order preserving bijection $\varphi: P \rightarrow Q$ such that

$$x \leq_P y \Leftrightarrow \varphi(x) \leq_Q \varphi(y).$$

A poset of the type $(\{a, b, c, d\}, < \mid a < c, b < c, b < d)$ is called an *N-poset* as its Hasse diagram looks like the letter *N*:

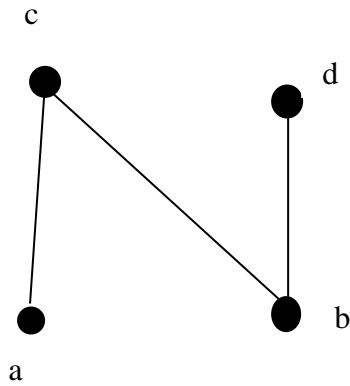


Figure 1

A poset (P, \leq_P) is called *N-free* if there exists no subposet A of P isomorphic to an N -poset.

Let (P, \leq_P) and (Q, \leq_Q) be two disjoint posets. The *disjoint (cardinal) sum* $P + Q$ is the poset $(P \cup Q, \leq_{P+Q})$ such that $x \leq_{P+Q} y$ if and only if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$. The *linear (ordinal) sum* $P \oplus Q$ is the poset $(P + Q, \leq_{P \oplus Q})$ such that $x \leq_{P \oplus Q} y$ if and only if $x \leq_{P+Q} y$ or $x \in P$ and $y \in Q$ with x preceding y . In other words, $P \oplus Q$ is obtained from $P + Q$ by adding $x \leq y$ (or x preceding y) for all $x \in P$ and $y \in Q$.

A k - antichain is defined as the disjoint union of k singletons.

A poset (P, \leq_P) is called *series-parallel* if it can be constructed from singletons by using disjoint union and linear sum.

A crown on $2n$ elements $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ is the partial order defined:

For all indices $i \neq j$, the elements a_i and a_j are incomparable, the elements b_i and b_j are incomparable, but $a_i < b_j$; and for each i , the elements a_i and b_i are incomparable.

A 4-crown (or a crown with four elements) poset is isomorphic to $2 \oplus 2$ (where 2 is a two elements antichain) and is a series parallel poset. The N -poset can be described as

$2 \oplus 2$ with one comparability missing. For example let $P = \{x, y\}$ and $Q = \{r, s\}$ be the two elements antichains. It is clear that $x <_{P \oplus Q} r, x <_{P \oplus Q} s, y <_{P \oplus Q} r, y <_{P \oplus Q} s$ for the poset $P \oplus Q$ (by definition).

The Hasse diagram follows:

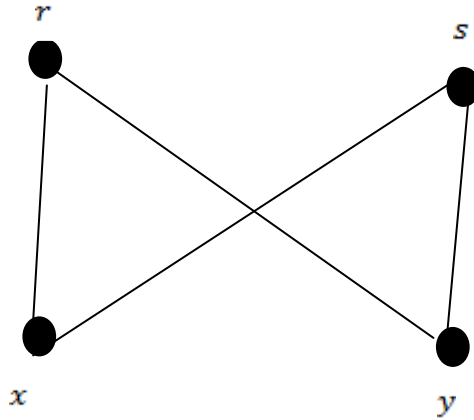


Figure 2

figure 2 above is an N poset Hasse diagram with the comparability $x <_{P \oplus Q} s$ missing.

Series-parallel posets can also be characterized as N -free posets (Valdes *et al.*, 1982)

A cycle is a poset (P, \leq_P) with Hasse diagram in figure 3(a) where $n \geq 2$

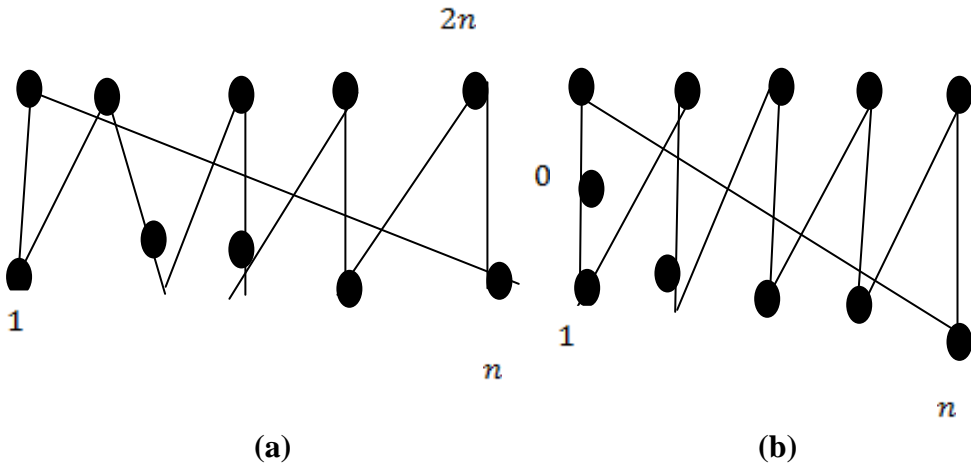


Figure 3 (a) Cycle (b) Crown

The young's lattice $L(m, n)$, where m, n are positive integers, is a poset defined on $\{(a_1, a_2, \dots, a_m) \mid 0 \leq a_1 \leq a_2 \leq \dots \leq a_m \leq n, a_i \in \mathbb{Z}\}$ with the order relation:

$(a_1, a_2, \dots, a_m) \leq (b_1, b_2, \dots, b_m)$ if and only if $a_i \leq b_i, i = 1, 2, \dots, m$.

The *height* of a poset (P, \leq_P) , denoted $h(P)$, is defined to be the cardinality of its longest chain. The width of (P, \leq_P) , denoted by $w(P)$, is defined to be the cardinality of its largest antichain. It is easy, though not trivial, to see that the following results hold

(Dilworth, 1950; Brualdi *et al.*, 1992):

$w(P)$ equals the minimum number of chains in a partition of P into chains, $h(P)$ equals the minimum number of antichains in a partition of P into antichains, $w(P) \geq w(P \setminus \{x\}) \geq w(P) - 1$ and

$h(P) \geq h(P \setminus \{x\}) \geq h(P) - 1$.

A Poset P is called *width-critical* if $w(P \setminus \{x\}) < w(P)$ and *height-critical*

if $h(P \setminus \{x\}) < h(P)$, for all $x \in P$.

It follows that a poset P is width-critical if and only if P is an antichain and height-critical if and only if P is totally ordered.

3. Algorithms for constructing linear extensions

3.1 Definitions of some basic terms and related results pertaining to ordered sets

Let (P, \leq_P) be a finite nonempty poset. A total ordering $<$ is said to be compatible with the partial ordering \leq_P if $a < b$ whenever $a \leq_P b$. The scheme for constructing a compatible total ordering from a partial ordering is

called *topological sorting* and the outcome is called a *topological sort* (or *linear extension*). In other words, a linear extension of P is a linear order which contains P .

Let L denote a linear extension of (P, \leq_P) and $L(P)$ denotes set of all linear extensions of P . $L(P)$ is nonempty for any P (Szpilrajn, 1930). That is, every order can be extended to a linear order. In fact, a stronger result has been proved: Let \leq_P be an order on P and let $x, y \in P$ such that $x \parallel y$. Then there exist two linear extensions \leq_{L_1} and \leq_{L_2} of \leq_P such that $x \leq_{L_1} y$ and $y \leq_{L_2} x$.

A linear extension L is said to *reverse* the incomparable pair (x, y) when $x \geq y$ in L . A family \mathcal{R} of linear extensions of P reverses $A \subseteq inc(P)$ if for every $(x, y) \in A$, there exists some $L \in \mathcal{R}$ such that $x \geq y$ in L .

The dual of a linear extension L of a poset (P, \leq_P) denoted L^d , is a linear order obtained by reversing the order of the linear extension L . The dual of a poset (P, \leq_P) , denoted P^d , is the poset obtained by reversing its order.

A consecutive pair (x_i, x_{i+1}) of elements in L is called a *jump* or *setup* of (P, \leq_P) in L if x_i and x_{i+1} are incomparable in (P, \leq_P) . We denote the number of jumps of (P, \leq_P) in L by $r(L, P)$. The *jump number* $r(P)$ of (P, \leq_P) is the minimum of $r(L, P)$ over all linear extensions L of (P, \leq_P) .

A Poset P is called *jump critical* if $r(P \setminus \{x\}) < r(P)$ for each $x \in P$.

A jump-critical Poset (P, \leq_p) with jump number m has atmost $(m + 1)!$ elements (El-Zahar & Schmerl, 1984), and there are precisely 17 jump-critical posets with jump number atmost 3 (El-Zahar & Rival, 1985). It is recognized that characterizing jump-critical posets turns out to be a considerably complicated problem. Pulleyblank proved that jump number problem viz. *schedule the tasks to minimize the number of jumps is NP-hard* (Bouchitte & Habib, 1987).

It follows from Dilworth’s theorem that $r(P) \geq w(P)$. If $r(P) = w(P)$, then (P, \leq_p) is called a *Dilworth poset* or simply a *D- poset* . It is shown that a poset which does not have a subposet isomorphic to a cycle in figure 2(a) is a *D- poset* (Duffus et al., 1982).

Syslo ((Bouchitte & Habib, 1987). put forward a polynomial algorithm to characterize Dilworth posets in the case where the antichain of maximal elements is a maximal-sized antichain. It is observed that the class of Dilworth posets does not seem to be nice with respect to computational complexity.

If $r(L, P) = r(P)$, then L is called an *optimal linear extension* of (P, \leq_p) . We denote the set of all optimal linear extensions of (P, \leq_p) by $\mathcal{O}(P)$.

3.2 Knuth’s (Bucket) sorting algorithm

Essentially, the topological sort of a finite partial order P is a linear

order a_1, a_2, \dots, a_n of elements of P such that $i < j$ whenever $a_i < a_j$ in P i.e; x precedes y in the partial ordering implies x precedes y in the linear extension (Knuth, 1973). The idea is to pick a minimal element and then to remove it from the poset, and continue the process with the truncated poset until it gets exhausted. A very fast algorithm and its implementation for computing a topological sort of a poset is presented in (Knuth, 1973) . As a matter of fact, this is a well-documented work on sorting. In its simplest form, the algorithm for constructing a total ordering in the finite poset $(P, <_p)$ can be depicted as below:

Since $(P, <_p)$ is *finite* and nonempty, it has minimal elements. Let a_1 be a minimal element i.e. a_1 is not preceded by any other object in the ordering $(P, <_p)$ which is chosen first. Again $(P - \{a_1\}, <_p)$ is also a poset. If it is non-empty, let a_2 be one of its minimal elements, which is chosen next and continue the process until no element remains to be further chosen. Since P is finite, this process must terminate and give rise to a sequence of elements a_1, a_2, \dots, a_n along with the desired total ordering defined by $a_1 < a_2 < \dots < a_n$. The compatibility of the total ordering $<$ with the original partial ordering $<_p$ can easily be verified i.e. by the definition given above, $a <_p b \Rightarrow a < b$ for all a, b in the ordering. It needs to be constantly

observed that $a < b$ only if a is chosen before b .

Alternatively, for a subset S of P , we denote the set of minimal elements of P restricted to S by $MIN(S)$. The algorithm *LIN* for computing a linear extension $[a_1, a_2, \dots, a_n]$ of the poset $(P, <_P)$ (Kierstead *et al.*, 1987) is defined:

ALGORITHM LIN:

SET $R = P$, $M = MIN(R)$

FOR $i = 0, \dots, n - 1$

CHOOSE $a_{i+1} \in M$

SET $R = R \setminus \{a_{i+1}\}$, $M = MIN(R)$

END

For any sequence of choices of the points a_{i+1} , the algorithm *LIN* produces a linear extension of $(P, <_P)$; and every linear extension of $(P, <_P)$ is obtained from *LIN* by a suitable sequence of choices of a_{i+1} .

Example 1:

Let $P = \{1, 2, 3, 4, 8, 12\}$ and the partial ordering relation be “divides” denoted by $|$. The scheme to find a compatible total ordering for the poset $(\{1, 2, 3, 4, 8, 12\}, |)$ may be outlined as follows:

At first stage, 1 is the only minimal element and hence gets selected. Next, we need to select a minimal element of $(\{2, 3, 4, 8, 12\}, |)$. At this stage, 2 and 3 are the two minimal elements, we select 3. Next, we need to select a minimal element of $(\{2, 4, 8, 12\}, |)$.

At this stage, 2 is the only minimal element. Next we need to select any minimal element of $(\{4, 8, 12\}, |)$. Here, 4 is the only minimal element. Next, as both 8 and 12 are minimal elements of $(\{8, 12\}, |)$, we select 12. Finally, 8 is left as the last element. The outcome is the total ordering

$1 < 3 < 2 < 4 < 12 < 8$. A linear representation of the above can be depicted as follows: $1 \rightarrow 3 \sim 2 \rightarrow 4 \rightarrow 12 \sim 8$. Another compatible total ordering for the same partial ordering may be constructed as follows: $1 < 3 < 2 < 4 < 8 < 12$. Hence it follows that a compatible total ordering for a given partial ordering may not be unique. In fact, the size of the family of linear extensions of a poset P varies from 1 (if P is a chain) to $n!$ (if P is an n -element antichain). Note, however, that it may not be even possible to construct such a total ordering if loops were present.

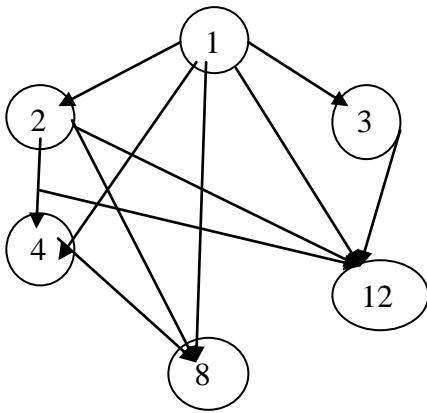
3.3 Depth first traversal algorithm

Another algorithm used for the computation of a topological sort is totally based on *depth first traversal* (Papamantou, 2004):

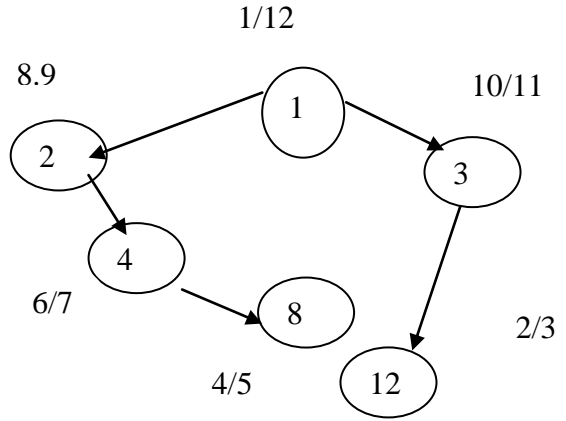
- (i) Perform DFS to compute the discovery/finishing times $d[v]/f[v]$ for each vertex v representing an element in the Hasse diagram of the poset.
- (ii) As each vertex is finished, insert it to the front of a linked list

- (iii) Return the linked list of vertices
- (iv) Output the vertices in reverse order of finishing time to get the topological sort of the poset (Skiena, 1997).

Example 2: Below is an original graph of the poset $P = (\{1,2,3,4,8,12\}, |)$ and its Depth first search (DFS) forest.



Original graph of poset P



Depth First Search (DFS) forest of poset P

Figure 4

Final order: $1 < 3 < 2 < 4 < 8 < 12$.

Note that the DFS could generate other distinct topological sorts using the same vertex 1.

4. Algorithm for constructing greedy linear extensions

A more restrictive class of linear extensions of a poset $(P, <_P)$ is obtained by further restricting the choice of a_{i+1} to generate topological sorts called *greedy linear extensions*. The algorithm for computing greedy linear extensions of a poset (Cogis & Habib, 1979;

Brualdi et al., 1992) is given as follows:

- (i) Choose a minimal element a_1 of P
- (ii) Suppose a_1, a_2, \dots, a_i have been chosen.

T_1 : If there is at least one minimal element of

$P \setminus \{a_1, a_2, \dots, a_i\}$ which covers a_i then choose a_{i+1} to be any such minimal

element; otherwise, choose a_{i+1} to be any minimal element of $P \setminus \{a_1, a_2, \dots, a_i\}$.

More precisely, a linear extension of $(P, <_P)$ is greedy if and only if it is obtained from the following algorithm **GREEDY** by a suitable sequence of choices of the points a_{i+1} (Kierstead et al., 1987):

ALGORITHM GREEDY

SET $R = P, M = \text{MIN}(R), G = M$

FOR $i = 0, \dots, n - 1$

CHOOSE $a_{i+1} \in G$

SET $R = R \setminus \{a_{i+1}\}, M = \text{MIN}(R)$

IF $M \cap U(a_{i+1}) \neq \emptyset$

THEN SET $G = M \cap U(a_{i+1})$

ELSE SET $G = M$

END

Example 3: Hasse diagram of the poset $(\{1,2,3,4,8,12\}, |)$ and its corresponding linear extensions:

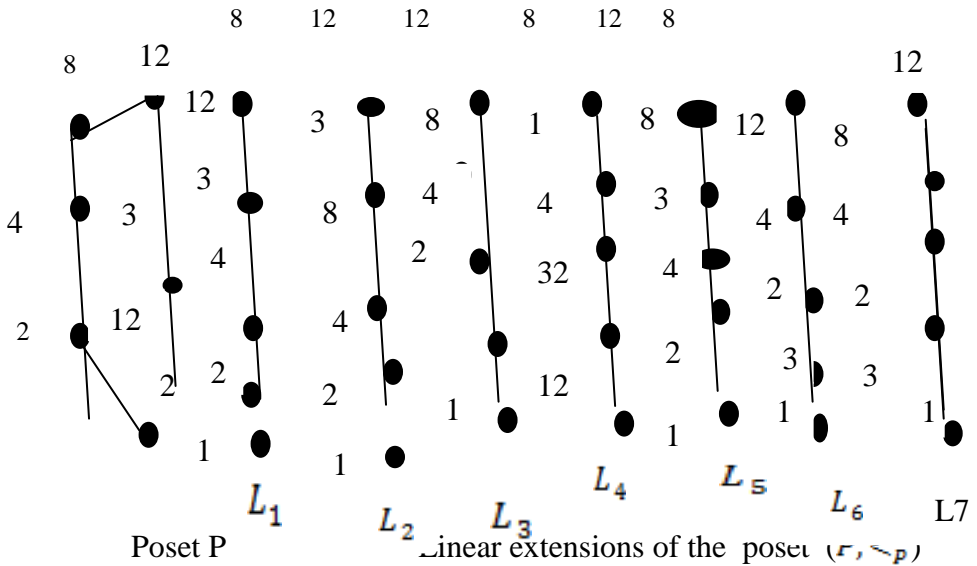


Figure 5

By definition, L_2, L_6 and L_7 are greedy linear extensions of the poset

P , but L_1, L_3, L_4 and L_5 are not greedy.

Let $\mathcal{G}(P)$ denote the set of all greedy linear extensions of the poset P . A poset (P, \leq_p) is *greedy* if $\mathcal{G}(P) \subseteq \mathcal{O}(P)$; that is, every greedy linear extension is optimal.

Every greedy linear extension is optimal for the jump number on the class of *series parallel* posets (Cogis & Habib, 1979). Every N -free poset is greedy (Rival, 1986). An optimal linear extension of Dilworth poset is necessarily greedy (Bouchitte & Habib, 1987).

The *Young's Lattice* $L(m, n)$ is greedy if and only if one of (1) $m = 1$ or $n = 1$ and (2) $m \leq 2$ and $n \leq 2$ is satisfied. Every poset (P, \leq_p) , containing no subposet isomorphic to figure 2(b) given in section 2, satisfies $\mathcal{O}(P) \subseteq \mathcal{G}(P)$ (El-Zahar & Rival, 1985).

A poset P is reversible if $L^d \in \mathcal{G}(P^d)$ whenever $L \in \mathcal{G}(P)$. A poset (P, \leq_p) is reversible if and only if $\mathcal{O}(P) = \mathcal{G}(P)$ (Rival & Zaguia, 1986; Jung, 1992).

5. Algorithm for constructing super greedy (depth-first greedy (dfgreedy)) linear extensions.

A further restrictive class of linear extensions of a poset (P, \leq_p) is the class of *super greedy* (*depth-first greedy* (dfgreedy)) linear extensions.

A greedy linear extension of (P, \leq_p) is *super greedy* if it is obtained by applying the following scheme (Bouchitte et al., 1985; Ducournau & Habib, 1987) :

(i) Choose for a_1 any minimal element of P

(ii) If $\{a_1, a_2, \dots, a_i\}$ is super greedy, then choose for a_{i+1} any minimal element of

$P \setminus \{a_1, a_2, \dots, a_i\}$ covering a_k , where $k \leq i$ is the greatest subscript, if there exists one; otherwise choose any minimal element of $P \setminus \{a_1, a_2, \dots, a_i\}$.

Alternatively, a linear extension L of (P, \leq_p) is super greedy if and only if it is obtained by applying the following algorithm *SUPER GREEDY* by a suitable sequence of choices of the points a_{i+1} (Kierstead et al., 1987):

ALGORITHM SUPER GREEDY

```

SET R = P, M = MIN(R), SG = M
FOR i = 0, ..., n - 1
  CHOOSE  $a_{i+1} \in SG$ 
  SET R = R \ { $a_{i+1}$ }, M = MIN(R), k = i
  WHILE  $M \cap U(a_k) = \emptyset$  AND  $k \neq 0$  DO
    SET k = k - 1
  IF k  $\neq 0$ ,
  THEN SET SG =  $M \cap U(a_k)$ 
END
    
```

In example 3, L_5 , L_6 and L_7 are super greedy linear extensions.

The notion of super greedy linear extension was introduced in (Pretzel, 1985), and studied some of its algorithmic properties studied (Bouchitte et al., 1985). Every super greedy linear extension is

greedy i.e; $\mathcal{SG}(P) \subseteq \mathcal{G}(P)$, where $\mathcal{SG}(P)$ denotes the set of all super greedy linear extensions of a poset (P, \leq_P) (Bouchitte & Habib, 1987). Computational complexity aspect of greedy and super greedy linear extension construction associated with the jump number has been studied (Kierstead, 1986).

6. Counting topological sorts, Dimension, and Realizers of a poset

The following are some established facts in this regard (Trotter, 1991; Brualdi et al., 1992; Skiena , 1997; Schroder, 2003; Kloch, 2007):

6.1 Counting topological sorts

- (i) Posets with no elements have exactly one linear extension, the null set.
- (ii) A Poset that is a chain has just one linear extension which is itself.
- (iii) A Poset that is an antichain of n elements has $n!$ linear extensions
- (iv) The number of linear extensions of all other Posets with n elements lies between two bounds mentioned in (ii) and (iii).

(v) $L(P) = \sum_k \text{minimal } L(P \setminus \{k\})$, where $L(P)$ denotes the number of all linear extensions of a given Poset P . The $L(P)$ for various linear extensions outlined above can be computed using the formulae viz;

$$L(P_{\text{greedy}}) = \sum_k \text{greedy minimal } L(P \setminus \{k\})$$

and

$$L(P_{\text{supgreedy}}) = \sum_k \text{super greedy minimal } L(P \setminus \{k\})$$

where

$L(P), L(P_{\text{greedy}})$ and $L(P_{\text{supgreedy}})$ denote the number of linear extensions, greedy linear extensions and super greedy linear extensions of a given Poset $(P, <_P)$ respectively. The expressions $L(P), L(P_{\text{greedy}}), L(P_{\text{supgreedy}})$ are also useful for enumerating all the linear extensions of each kind.

Example 4. We enumerate $L(P), L(P_{\text{greedy}})$ and $L(P_{\text{supgreedy}})$ of the poset $(P, <_P)$ given by the Hasse diagram below:

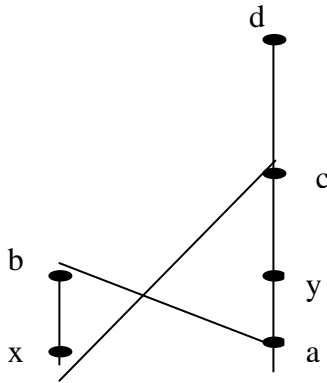


Figure 6

Let $L^{xy} \dots$ denote a shorthand notation for $L(P \setminus \{x, y, \dots\})$. We have the following:

$$\begin{aligned}
 L(P) &= L^a + L^x \\
 &= (L^{ax} + L^{ay}) + L^{xa} \\
 &= (L^{axb} + L^{axy}) + L^{ayx} + (L^{xab} + L^{xay}) \\
 &= L^{axby} + (L^{axyb} + L^{axyc}) + (L^{ayxb} + L^{ayxc}) + L^{xaby} + (L^{xayb} + L^{xayc}) \\
 &= L^{axbyc} + L^{axybc} + (L^{axycb} + L^{axycd}) + L^{ayxbc} + (L^{ayxcb} + L^{ayxcd}) + \\
 &\quad L^{xabyc} + (L^{xaycb} + L^{xaycd}) \\
 &= L^{axbycd} + L^{axybcd} + L^{axycbd} + L^{axycdb} + L^{ayxbcd} + L^{ayxcbd} + L^{ayxcdb} \\
 &\quad + L^{xabycd} + L^{xaybcd} + L^{xaycbd} + L^{xaycdb} . \\
 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 11
 \end{aligned}$$

Corresponding to the linear extensions $a < x < b < y < c < d$,

$a < x < y < b < c < d, a < x < y < c < b < d, a < x < y < c < d < b,$
 $a < y < x < b < c < d, a < y < x < c < b < d, a < y < x < c < d < b$
 $x < a < b < y < c < d, x < a < y < b < c < d, x < a < y < c < b < d,$
 $x < a < y < c < d < b$, respectively.

$$\begin{aligned}
 L(P_{\text{greedy}}) &= L^a + L^x \\
 &= L^{ay} + L^{xa} \\
 &= L^{ayx} + L^{xay} + L^{xab} \\
 &= L^{ayxb} + L^{ayxc} + L^{xayc} + L^{xab y} \\
 &= L^{ayxbc} + L^{ayxcb} + L^{xaycb} + L^{xaycd} + L^{xab y c} \\
 \\
 &= L^{ayxbcd} + L^{ayxc b d} + L^{xayc b d} + L^{xayc d b} + L^{xab y c d} \\
 &= 1 + 1 + 1 + 1 + 1 = 5
 \end{aligned}$$

corresponding to the greedy linear extensions $a < y < x < b < c < d$,

$a < y < x < c < b < d, x < a < y < c < b < d, x < a < y < c < d < b$,
and

$x < a < b < y < c < d$, respectively.

$$\begin{aligned}
 L(P_{\text{supgreedy}}) &= L^a + L^x \\
 &= L^{ay} + L^{xa} \\
 &= L^{ayx} + L^{xay} + L^{xab} \\
 &= L^{ayxb} + L^{ayxc} + L^{xayc} + L^{xab y} \\
 &= L^{ayxbc} + L^{ayxcd} + L^{xaycd} + L^{xab y c} \\
 &= L^{ayxbcd} + L^{ayxc b d} + L^{xayc d b} + L^{xab y c d}
 \end{aligned}$$

$= 1 + 1 + 1 + 1 = 4$ corresponding to the super greedy linear extensions

$a < y < x < b < c < d, a < y < x < c < d < b, x < a < y < c < d < b$
and

$x < a < b < y < c < d$,
respectively.

6.2 Realizers of a poset

Szpilrajn [33] proved that any order relation is the intersection of its linear extensions. In fact, not very infrequently, the intersection of only a few linear extensions of a given

Poset turns out to be sufficient to give rise to the original Poset. That is, for any poset P , there exists a finite set of its linear extensions which realizes P . This leads to the following definition:

If \mathcal{R} is a family of linear extensions of the poset (P, \leq_P) whose intersection is the order relation \leq_P ,

then \mathcal{R} is called a *realizer* of \leq_P . If $\mathcal{R} = \{L_1, L_2, \dots, L_n\}$ is a realizer of the poset (P, \leq_P) , then for any $A \subseteq P$, the set $A_{\mathcal{R}} = \{L_1|_A, L_2|_A, \dots, L_n|_A\}$ is a realizer of the subposet (A, \leq_P) .

The set of all greedy linear extensions of a poset (P, \leq_P) is a realizer called the *greedy realizer* and the family of greedy realizers is nonempty (Bouchitte et al., 1985). There exists a super greedy realizer for every ordered set (Kierstead et al., 1987).

6.3 Dimension of a poset

The dimension of a poset (P, \leq_P) , denoted $\dim(P)$, is defined as the minimum cardinality of a realizer for the poset (P, \leq_P) (Dushnik & Miller, 1941). In other words, the dimension of a poset is the minimal number of its linear extensions whose intersection is the original poset. That is, $\dim(P)$ is the least positive integer k for which there exists a family $\mathcal{R} = \{L_1, L_2, \dots, L_k\}$ of linear extensions of P such that $\bigcap \mathcal{R} = \bigcap_{i=1}^k L_i = P$. It follows from Dilworth's theorem that the dimension of an order never exceeds its width i.e., $\dim(P) \leq w(P)$. Also, it follows from the definition, that the removal of a point from P cannot increase its dimension but it can decrease by at most one (Hiraguchi, 1951). Thus we have $\dim P \geq \dim \{P \setminus \{x\}\} \geq \dim P - 1$ for all $x \in P$. Tree-shaped posets are 2-dimensional. In general, for a poset (P, \leq_P) with $|P| = n$, its

upper bound is given by $\dim P \leq \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, if $A \subseteq P$, then $\dim(A, \leq_P) \leq n$.

A Poset (P, \leq_P) (of dimension d) is called *dimension-critical* (or *d-irreducible*) provided $\dim(P \setminus \{x\}) < \dim P, \forall x \in P$. In other words, P is *d-irreducible* if it has a dimension $d \geq 2$ and the removal of any element lowers its dimension. The *3-irreducible* Posets have been characterized, but no characterization is known for the *d-irreducible* posets for $d \geq 4$. However, it is known that for each $d \geq 3$, there exist infinitely many dimension-critical posets of dimension d (Kelley, 1977; Trotter & Moore, 1976). It is shown that the computation of dimension itself is an *NP*-complete problem. In particular, it is polynomial time solvable if a partial order has dimension at most 2, but the case for having dimension at most 3 is *NP*-complete (Yannakakis, 1982). However, whether the jump number is *NP*-complete for the particular class of two dimensional posets is still an open problem (Bouchitte & Habib, 1987).

The notion of greedy dimension of a poset is studied in (Bouchitte et al., 1985). It is observed that the existence of a *greedy realizer* and thus of the *greedy dimension* immediately follows from a result obtained in (El-Zahar & Rival, 1985) that for every incomparable pair (a, b) , there exists a greedy linear extension $<_{\mathcal{R}}$ with $a <_{\mathcal{R}} b$.

This is proved by demonstrating algorithmically that such a greedy linear extension exists for every critical pair. Further, in course of studying the relationship between the greedy dimension and the ordinary dimension of a poset, the existence of equality between them for a wide range of posets, including the N -free posets, two dimensional posets and distributive lattices has been proved.

Following the definition of $\text{crit}(P)$ for a poset (P, \leq_P) , a family \mathcal{R} of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \text{crit}(P)$, there exists some $L \in \mathcal{R}$ such that $x \geq y$ in L , and hence the dimension of P is just the least integer t for which there exists a family \mathcal{R} of linear extension of P which reverses $\text{crit}(P)$ (Trotter, 1991).

Furthermore, if $\dim P = 2$, then every minimal realizer of (P, \leq_P) is greedy.

Since $\text{sg}(P) \subseteq \mathcal{G}(P)$, we note here that

$$\dim(P) \leq \dim_{\mathcal{G}}(P) \leq \dim_{\text{sg}}(P)$$

holds

(Kierstead & Trotter, 1985).

Besides a wealth of results related to bounds for ordinary and greedy dimensions of a poset, the best possible upper bounds for the super greedy dimension of a poset (P, \leq_P) in terms of $|P - A|$ and width $(P - A)$, where A is a maximal antichain has been proved (Kierstead et al., 1987). Summarily, we have the following:

Removing one point from a poset does not increase any of the parameters: width, height, jump number and dimension. However, it can decrease each of them by atmost one.

If one comparability pair is removed from a poset, its result is a poset in general. However, if only a comparability pair which cannot be recovered by transitivity is removed, the result is still a poset. Thus, removing the comparability $x < y$ results in a poset if and only if y covers x .

Similarly, the addition of one comparability pair does not in general results in a poset. However, if only a comparability which does not *force* other comparabilities is added, the result is again a poset. In other words, the comparability $x < y$ can be added to P with the result being a poset (with exactly one more comparability) if and only if $u < x$ in P implies $u < y$, and $y < v$ in P implies $x < v$. Such a pair (x, y) is called an *nonforcing ordered pair* of P (Rabinovitch & Rival, 1979).

Despite the emergence of consequences that posets exist with bounded height but arbitrary large dimension (Trotter, 1991), numerous significant contributions towards characterizing the dimension parameter of a poset are around (Kelley & Trotter, 1982).

7. Some future directions

(i) In face of the fact that every poset has a greedy optimal linear extension, the characterization for

the existence of non-greedy optimal linear extension of a poset need to be investigated.

(ii) The choice of some useful characterization (say, *stability*, etc.) of a poset in terms of the size of its realizers versus the size of the class of its all linear extensions could be investigated further.

(iii) Many nice properties of realizers are known, but how to compute them effectively needs further vindication.

(iv) A number of optimization problems need to be addressed; for example, constructing an efficient algorithm to compute a linear extension that minimizes the number of jumps (Bouchitte & Habib, 1987).

(v) Studies related to various concepts described in this paper on a finite multiset are yet to be conducted (Anderson, 1987; Girish & Sunil, 2009).

(vi) Multiset as a model for *multi-attribute objects* used in discovering intelligent systems, control of Non-linear mechanical systems etc. may get simplified by using topological sorting (Petrovsky, 1997).

(vii) Topological sorting can also be used in discovering computing simulators for biological systems (Krishnamurthy, 2005).

(viii) Last but not the least, some *open* problems, like $\frac{1}{2} - \frac{2}{3}$ problem need our attention (Felsner & Trotter, 1993).

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